Cubics passing through the Foci of an Inscribed Conic

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Created : January 21, 2006
Last update : February 27, 2015

Abstract

We study the circum-cubics which pass through the four foci of any conic inscribed in the reference triangle ABC.

Several usual transformations are very frequent in this paper. We will use the following notations :
– $\mathbf{g}P$ = isogonal conjugate of P,
– $\mathbf{t}P$ = isotomic conjugate of P,
– $\mathbf{c}P$ = complement of P,
– $\mathbf{a}P$ = anticomplement of P.

The reference numbers for cubics of the form Knnn are those of [2].
The vocabulary and notations we use for cubics are those of [1].

1 Inconics

We start with several lemmas.

1.1 Axes

Let $Q = p : q : r$ be a finite point not lying on a sideline of the reference triangle ABC and let $\mathcal{IC}(Q)$ be the inscribed conic (inconic) with center Q. We shall also suppose that Q is not an in/excenter so that $\mathcal{IC}(Q)$ is not a parabola nor a circle to make sure that it has four distinct finite foci.

Recall that the perspector P of $\mathcal{IC}(Q)$ is $\mathbf{ta}Q$ namely $P = \dfrac{1}{q+r-p} ::$ and that the equation of $\mathcal{IC}(Q)$ is

$$\sum_{\text{cyclic}} (q+r-p)^2\, x^2 - 2(p+q-r)(r+p-q)yz = 0.$$

Lemma 1. *$\mathcal{IC}(Q)$ has the same points at infinity as $\mathcal{CP}(Q^2)$, the circumconic with perspector the barycentric square Q^2 of Q. In other words, $\mathcal{IC}(Q)$ and $\mathcal{CP}(Q^2)$ are homothetic.*

The equation of $\mathcal{CP}(Q^2)$ is : $p^2\, y\, z + q^2\, z\, x + r^2\, x\, y = 0$. Its center is $p^2(q^2 + r^2 - p^2) ::$, the cevian quotient of G and Q^2.

This barycentric square Q^2 is either

- the pole of the centroid G of ABC in the pencil $\mathcal{F}(Q)$ of conics passing through Q and the vertices Q_a, Q_b, Q_c of the anticevian triangle of Q,
- the intersection of the lines G, $\mathbf{tc}Q$ and Q, $\mathbf{ct}Q$,
- the trilinear pole of the line passing through the midpoints of a vertex of the cevian triangle of Q and the foot of the trilinear polar of Q on the same sideline of ABC.

Lemma 2. *The axes of $\mathcal{IC}(Q)$ are the parallels at Q to the asymptotes of the diagonal rectangular hyperbola $\mathcal{H}_d$ passing through the in/excenters and Q. See figure 1.*

The equation of $\mathcal{H}_d$ is

$$\sum_{\text{cyclic}} (c^2q^2 - b^2r^2)\, x^2 = 0$$

and the equation of the axes is

$$\sum_{\text{cyclic}} (q + r - p)(c^2q^2 - b^2r^2)\, x^2 + 2[p^2(c^2q - b^2r) + a^2(q - r)qr]yz = 0.$$

Note that $\mathcal{H}_d$ is a member of the pencil $\mathcal{F}(Q)$ cited above. It is also the locus of centers of the inconics having their axes parallel to those of $\mathcal{IC}(Q)$.

Let Ω_d be the center of $\mathcal{H}_d$ and S_d be the fourth intersection of $\mathcal{H}_d$ and the circle through the excenters.

Ω_d obviously lies on the circumcircle and is the midpoint of the incenter and S_d.

Lemma 3. *Ω_d is also the trilinear pole of the line KQ^2.*

The isogonal conjugate $\mathbf{g}\Omega_d$ of Ω_d is the infinite point of the trilinear polar of $\mathbf{g}Q^2$, the isogonal conjugate of the barycentric square of Q.

We have :

$$\Omega_d = \frac{1}{c^2q^2 - b^2r^2} :: \quad \text{and} \quad \mathbf{g}\Omega_d = \frac{q^2}{b^2} - \frac{r^2}{c^2} :: .$$

Let $P_c = \mathbf{ct}\Omega_d$ (P_c lies on the orthic axis) and let $\Omega_c = \mathbf{c}\Omega_d$ (on the nine point circle). These points are :

$$P_c = (b^2 - c^2)p^2 - a^2(q^2 - r^2) ::,$$
$$\Omega_c = (c^2q^2 - b^2r^2)[(b^2 - c^2)p^2 - a^2(q^2 - r^2)] :: .$$

Lemma 4. *The axes of $\mathcal{IC}(Q)$ are the parallels at Q to the asymptotes of the rectangular circum-hyperbola $\mathcal{H}_c$ with center Ω_c and perspector P_c. See figure 1.*

The equation of $\mathcal{H}_c$ is :

$$\sum_{\text{cyclic}} [(b^2 - c^2)p^2 - a^2(q^2 - r^2)]\, yz = 0.$$

Remarks :

– $\mathcal{H}_c$ meets the circumcircle again at H', reflection of Ω_d in O or reflection of H in Ω_c.

– the isogonal transform of $\mathcal{H}_c$ is the line $\mathcal{L}_c$ parallel at O to the Simson line of Ω_d. This parallel meets the circumcircle at two points lying on the asymptotes of $\mathcal{H}_d$.

– $\mathcal{H}_d$ and $\mathcal{H}_c$ have two common points at infinity (hence they are homothetic) and two other finite (not always real) points on the line KQ^2. The midpoint of these two points lies on the line $\Omega_c\Omega_d$.

From this we can deduce a possible construction of these axes : $\mathcal{L}_c$ meets the circumcircle at two points E_1, E_2. The axes of $\mathcal{IC}(Q)$ are the parallels at Q to the lines Ω_dE_1 and Ω_dE_2.

Lemma 5. *The circumconics having their axes parallel to those of $\mathcal{IC}(Q)$ are the members of the pencil of circumconics passing through Ω_d.*

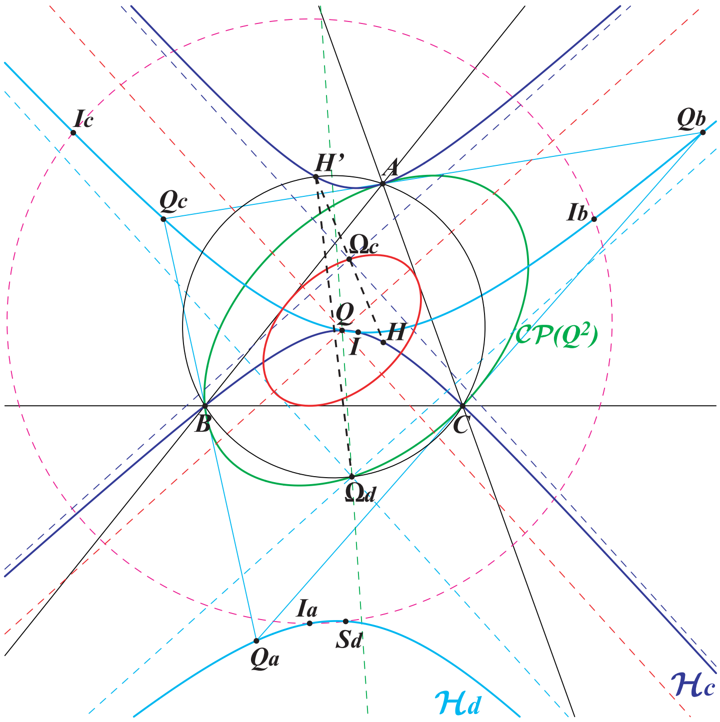

Figure 1: Axes of $\mathcal{IC}(Q)$

All these circumconics are centered on the rectangular hyperbola passing through O, the midpoints of ABC, the vertices of the cevian triangle of Ω_d. This rectangular hyperbola is the complement of $\mathcal{H}_c$.

The perspectors of these circumconics lie on the line KQ^2.

This pencil contains the circumcircle and a rectangular circum-hyperbola whose perspector must be the intersection of the line KQ^2 and the orthic axis. This hyperbola obviously contains H and Ω_d. Its equation is

$$\left(\sum_{\text{cyclic}} S_A p^2\right)\left(\sum_{\text{cyclic}} a^2yz\right) - 8\Delta^2\left(\sum_{\text{cyclic}} p^2yz\right) = 0,$$

where Δ is the area of ABC.

Lemma 6. *The axes of $\mathcal{IC}(Q)$ are the bisectors of the line passing through Q and* $\mathbf{g}Q$*, and the line passing through Q and the isogonal conjugate of the infinite point of the previous line.*

This latter lemma gives a simple construction of these axes which we use in the sequel.

1.2 Foci of inscribed conics

1.2.1 A classical construction

Recall that the foci of a conic are the fixed points on the axes of the involution which swaps the tangent and the normal at any point on the conic.

Let $P_aP_bP_c$ be the cevian triangle of the perspector P of $\mathcal{IC}(Q)$. These are the points of tangency of $\mathcal{IC}(Q)$ with the sidelines of ABC.

The sideline BC and the perpendicular at P_a to BC [1] meet one axis of $\mathcal{IC}(Q)$ at A_1, A'_1 respectively and the other at A_2, A'_2 respectively.

The circle centered at Q which is orthogonal to the circle with diameter $A_iA'_i$ meets the corresponding axis at two foci of $\mathcal{IC}(Q)$. Two of these foci are real (on the focal axis) and two are imaginary (on the non-focal axis). It is well known that they are two by two isogonal conjugates. See figure 2.

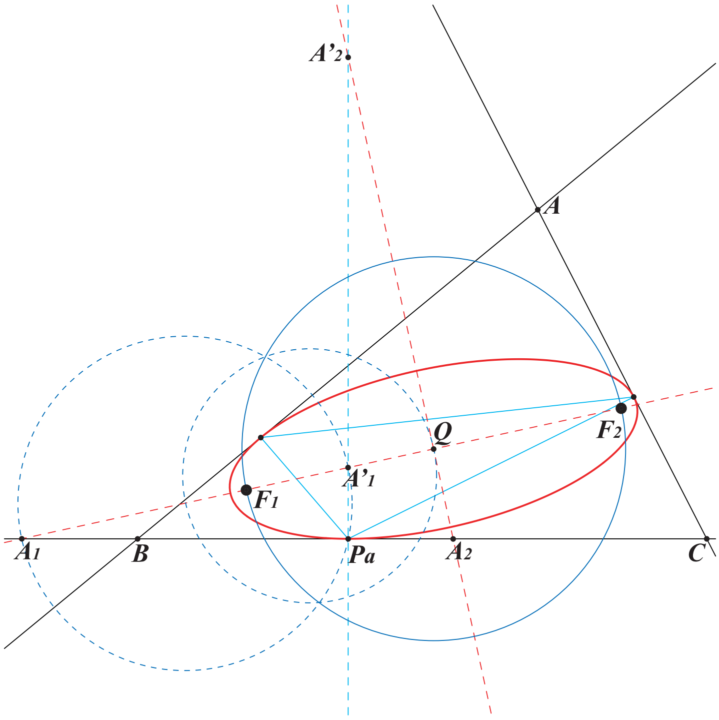

Figure 2: Foci of $\mathcal{IC}(Q)$

1.2.2 A simple construction with diagonal hyperbolas

We denote by $\mathcal{H}'_d$ the diagonal rectangular hyperbola which contains the in/excenters and $\mathbf{g}Q = Q^*$. This also contains the vertices of the anticevian triangle of $\mathbf{g}Q$.

$\mathcal{H}_d$ and $\mathcal{H}'_d$ are the two hyperbolas of the pencil they generate which are tangent to the line $Q\mathbf{g}Q$.

Theorem 1. *The foci of $\mathcal{IC}(Q)$ are the intersections of its axes and the rectangular hyperbola $\mathcal{H}_{Q^*}$ passing through $\mathbf{g}Q$ whose asymptotes are the parallels at Q to those of $\mathcal{H}'_d$.*

On the figure 3, ∞^* denotes the isogonal conjugate of the infinite point of the line $Q\mathbf{g}Q$ and E_1, E_2 are its reflections in the axes. The circles of the inversions with pole Q swapping $\mathbf{g}Q$ and E_1, E_2 contain the foci. Naturally, one of them only is real and contains the real foci F_1, F_2.

1.2.3 Other remarkable hyperbolas

It is obvious that the conics passing through the four foci of $\mathcal{IC}(Q)$ are rectangular hyperbolas forming a pencil generated by the axes of $\mathcal{IC}(Q)$ and $\mathcal{H}_{Q^*}$ for example.

This pencil contains three very simple hyperbolas $\mathcal{H}_A$, $\mathcal{H}_B$, $\mathcal{H}_C$ which are those passing through A, B, C respectively.

[1] These two lines are the tangent and the normal at P_a to the inconic.

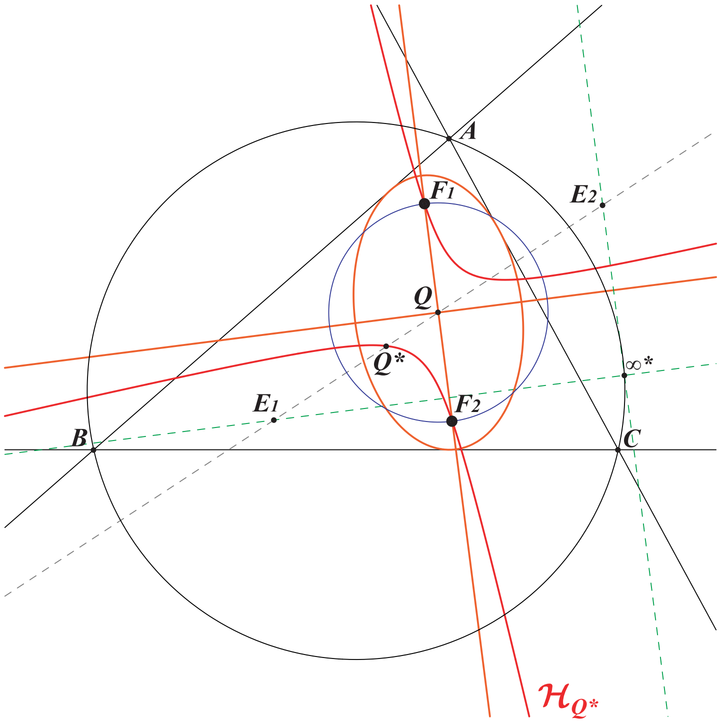

Figure 3: Construction of the Foci of $\mathcal{IC}(Q)$

Theorem 2. *$\mathcal{H}_A$ is the rectangular hyperbola passing through A whose asymptotes are the parallels at Q to the bisectors at A of the triangle ABC.*[2]

$\mathcal{H}_A$ has center Q and contains :
– A (the tangent at A to $\mathcal{H}_A$ passes through Q^*),
– the reflection A' of A in Q,
– the intersection A_b of AB with the reflection of AC in Q,
– the intersection A_c of AC with the reflection of AB in Q,
– the four foci of $\mathcal{IC}(Q)$. See figure 4.
$\mathcal{H}_A$ has equation

$$(p+q+r)(b^2z^2 - c^2y^2) + 2(x+y+z)(c^2qy - b^2rz) = 0.$$

2 Cubics and Foci of Inconics

We now consider the same inconic $\mathcal{IC}(Q)$ and call F_1, F_2 the real foci, F_1', F_2' the imaginary foci. Most of the time, these points have very complicated coordinates involving square roots and we shall not even try (although possible) to compute them.

We want to find all the circum-cubics passing through the four foci of $\mathcal{IC}(Q)$. Since all these cubics contain seven fixed points, they all belong to a same net $\mathcal{N}(Q)$ of cubics. This net is clearly stable under isogonal conjugation i.e. the isogonal transform of any cubic of $\mathcal{N}(Q)$ is another (not necessarily distinct) cubic of $\mathcal{N}(Q)$. Hence we need three independent cubics to generate the net.

[2]These bisectors are also those of the lines AQ, AQ^*.

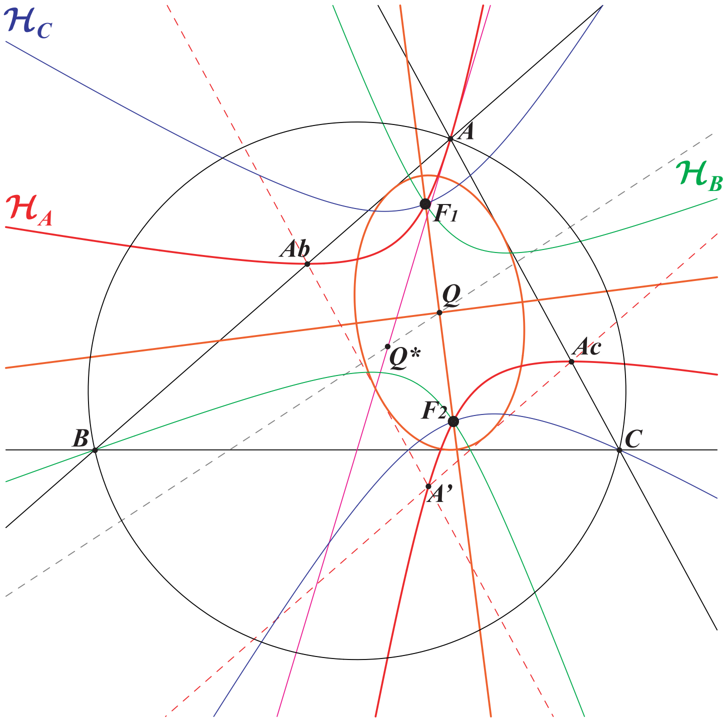

Figure 4: The hyperbolas $\mathcal{H}_A$, $\mathcal{H}_B$, $\mathcal{H}_C$

2.1 Three singular cubics generating the net

Let $\mathcal{K}_A$ be the degenerated cubic which is the union of $\mathcal{H}_A$ and the sideline BC. $\mathcal{K}_B$ and $\mathcal{K}_C$ are defined similarly.

These three cubics are three independent circumcubics passing through the foci of $\mathcal{IC}(Q)$, hence each cubic of $\mathcal{N}(Q)$ can be written as a linear combination of $\mathcal{K}_A$, $\mathcal{K}_B$ and $\mathcal{K}_C$. In another words, for any cubic of $\mathcal{N}(Q)$, one can find a point $P = u : v : w$ such that the cubic $\mathcal{K}_P$ belongs to $\mathcal{N}(Q)$ where

$$\mathcal{K}_P = u\,\mathcal{K}_A + v\,\mathcal{K}_B + w\,\mathcal{K}_C,$$

in which case the equation of any circumcubic passing through the foci of $\mathcal{IC}(Q)$ must take the form

$$\sum_{\text{cyclic}} u\,x\,[(p+q+r)(b^2z^2 - c^2y^2) + 2(x+y+z)(c^2qy - b^2rz)] = 0.$$

The isogonal transform of $\mathcal{K}_P$ is $\mathcal{K}_{P'}$ where P' is the reflection of P in Q. It follows that $\mathcal{K}_P$ is invariant under isogonal conjugation if and only if $P = Q$ or P lies at infinity. This is examined in the next section.

Construction of $\mathcal{K}_P$

One can easily verify that $\mathcal{K}_P$ is actually $sp\mathcal{K}(P, Q)$ as in CL055 and this gives the following construction of $\mathcal{K}_P$.

A variable line (L) passing through P is reflected about Q to give the line (L'). The isogonal transform of (L) meets (L') at two points on $\mathcal{K}_P$.

2.2 Special isogonal cubics of the net

This net $\mathcal{N}(Q)$ contains :

1. One and only one pivotal isogonal cubic $p\mathcal{K}$ whose pivot is Q. This is $\mathcal{K}_Q$ with equation

$$\sum_{\text{cyclic}} p\, x\, (c^2y^2 - b^2z^2) = 0.$$

   Note that $p\mathcal{K}$ cannot be a circular cubic since Q does not lie at infinity.

2. A family of cubics $ps\mathcal{K}$, see [4], (which contains the $p\mathcal{K}$ above) when P lies on the central cubic $p\mathcal{K}(G/Q, aaQ)$ with center Q. In particular, since this latter cubic contains aQ and aaQ with midpoint Q, the two cubics $\mathcal{K}_{aP}$ and $\mathcal{K}_{aaP}$ are two $ps\mathcal{K}$s of the family, one being the isogonal transform of the other.

3. A pencil of non-pivotal isogonal cubics $n\mathcal{K}$ obtained when P is an infinite point. Each cubic is the locus of foci of the inconics centered on a line $\mathcal{L}(Q)$ passing through Q.

   All these $n\mathcal{K}$ are circular focal cubics with singular focus on the circumcircle. Their roots lie on the trilinear polar of P, the perspector of $\mathcal{IC}(Q)$. This line has equation

$$\sum_{\text{cyclic}} (q + r - p)\, x = 0.$$

   Any two distinct such cubics $n\mathcal{K}$ generate the pencil and, consequently, these two cubics and the $p\mathcal{K}$ can also generate the net.

   It is convenient to choose two of the simple following $n\mathcal{K}$:

   - the unique $n\mathcal{K}_0$ of the pencil obtained when the line $\mathcal{L}(Q)$ contains the Lemoine point K. This gives the cubic with equation :

$$\sum_{\text{cyclic}} [(b^2 - c^2)p - a^2(q - r)]\, x\, (c^2y^2 + b^2z^2) = 0,$$

     passing through the infinite point of the line KQ, its isogonal conjugate (which is the singular focus) and naturally the four foci of the inconic with center K. The root of $n\mathcal{K}_0$ is the point $(b^2 - c^2)p - a^2(q - r) :: $, namely the complement of the isotomic conjugate of the tripole of the line KQ.

   - the cubic $n\mathcal{K}_\infty$ which has its root at infinity obtained when the line $\mathcal{L}(Q)$ contains the centroid G. This gives the cubic with equation :

$$\sum_{\text{cyclic}} (q - r)\, x\, (c^2y^2 + b^2z^2) + 2 \left(\sum_{\text{cyclic}} (b^2 - c^2)p \right) xyz = 0,$$

     passing through the infinite point of the line GQ, its isogonal conjugate (which is the singular focus) and naturally the four foci of the Steiner in-ellipse.

   Remark : these two cubics are defined and are distinct for any Q not lying on the line GK. In the case of Q on GK, we replace $n\mathcal{K}_\infty$ by

- the cubic $c\mathcal{K}$ obtained when the line $\mathcal{L}(Q)$ contains the incenter I. This gives the cubic with equation :

$$\sum_{\text{cyclic}} [(b-c)p - a(q-r)]\, x\,(c^2y^2 + b^2z^2) - 2\left(\sum_{\text{cyclic}} bc(b-c)p\right) xyz = 0,$$

which is a strophoid with node at I, passing through the infinite point of the line IQ, its isogonal conjugate (which is the singular focus).

Before we seek other interesting cubics of the net, we give three examples with $Q = G$, $Q = X_{39}$ (Brocard midpoint) and $Q = X_9$ (Mittenpunkt).

2.3 Examples

2.3.1 $\mathcal{IC}(Q)$ is the Steiner in-ellipse

In the case $Q = G$,

- $\mathcal{IC}(Q)$ is the Steiner in-ellipse,
- $p\mathcal{K}$ is the Thomson cubic K002,
- $n\mathcal{K}_0$ is the second Brocard cubic K018,
- $n\mathcal{K}_\infty$ is not defined,
- $c\mathcal{K}$ is the Gergonne strophoid K086.

2.3.2 $\mathcal{IC}(Q)$ is the Brocard ellipse

In the case $Q = X_{39}$,

- $\mathcal{IC}(Q)$ is the Brocard ellipse with real foci the Brocard points Ω_1, Ω_2,
- $p\mathcal{K}$ is the tenth Brocard cubic K326,
- $n\mathcal{K}_0$ is the third Brocard cubic K019,
- $n\mathcal{K}_\infty$ is the Brocard-Steiner focal cubic K248,
- $c\mathcal{K}$ is the Brocard strophoid K359. See figure 5.

2.3.3 $\mathcal{IC}(Q)$ is the Mandart ellipse

In the case $Q = X_9$,

- $\mathcal{IC}(Q)$ is the Mandart ellipse (see [3]),
- $p\mathcal{K}$ is K351,
- $n\mathcal{K}_0$ is the Pelletier strophoid K040,
- $n\mathcal{K}_\infty$ is the Mandart-Steiner focal cubic K352,
- $c\mathcal{K}$ is the Pelletier strophoid again.

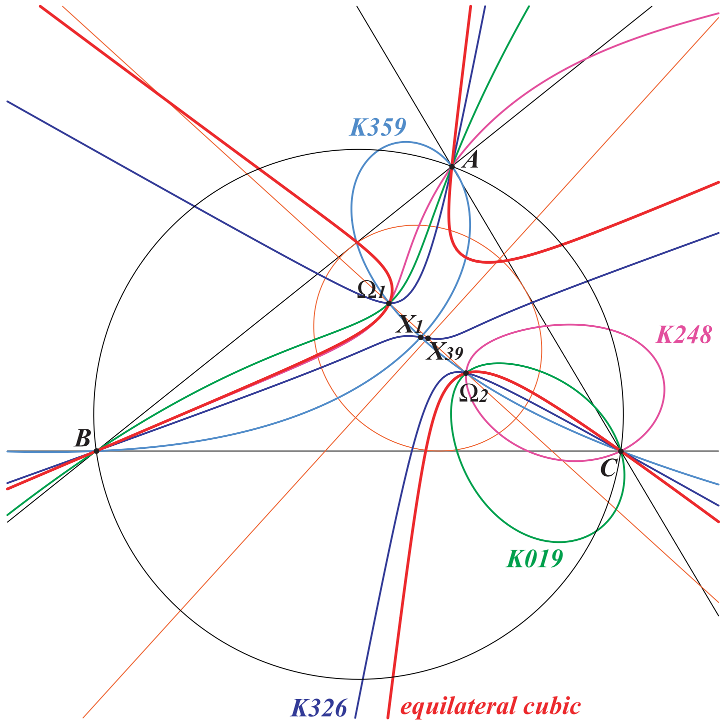

Figure 5: Cubics through the foci of the Brocard ellipse

2.4 Other cubics of the net

For a given Q, the center of $\mathcal{IC}(Q)$, the net of cubics contains some other interesting cubics, although, in some cases, these might coincide with those already mentioned or between themselves.

2.4.1 Equilateral cubics

The net contains one equilateral cubic $\mathcal{K}^{+}_{60}(Q)$ with equation

$$2(x+y+z)\sum_{\text{cyclic}} a^2(c^2S_C\,q - b^2S_B\,r)yz$$

$$= (p+q+r)\sum_{\text{cyclic}} a^2(c^2S_C\,y - b^2S_B\,z)yz,$$

showing it has always three real concurring asymptotes parallel to those of the McCay cubic and meets the McCay cubic again at the same points as the rectangular circum-hyperbola passing through **g**Q. Hence, $\mathcal{K}^{+}_{60}(Q)$ always passes through H and two other (not always real) points which are the isogonal conjugates of the intersections (other than O) of the McCay cubic and the line OQ.

The asymptotes concur at the point X such that $\overrightarrow{HX} = \frac{2}{3}\,\overrightarrow{HQ}$. The common points with the circumcircle are those of the pivotal isogonal cubic with pivot the reflection of O in Q.

The cubic also contains X (hence becomes a $\mathcal{K}^{++}_{60}(Q)$) if and only if Q lies on a central equilateral non-circum-cubic passing through H, X_{140}, X_{550} which is the homothetic of the two cubics $\mathcal{K}^{++}_{60}(X_{140})$ and $\mathcal{K}^{++}_{60}(X_{550})$. In this case, it is a central cubic.

For example, $\mathcal{K}^{+}_{60}(G)$ passes through the foci of the Steiner in-ellipse and $\mathcal{K}^{++}_{60}(H)$ is a central cubic passing through the foci of the in-conic with center H, perpector X_{253}.

2.4.2 Cubics passing through Q

Q always lie on $p\mathcal{K}$ (it is its pivot) but generally does not lie on the three already mentioned cubics with the following notable exceptions :

- Q lies on $n\mathcal{K}_0$ if and only if it is a point on the Grebe cubic K102,
- Q lies on $n\mathcal{K}_\infty$ if and only if it is a point on the Thomson cubic K002,
- Q lies on $c\mathcal{K}$ if and only if it is a point on the internal bisectors of ABC but, in this case, the cubic degenerates.

Thus, when Q is not a point on these curves, there is another cubic denoted by $n\mathcal{K}_Q$ which passes through the foci and the center Q of the inconic. It also contains the isogonal conjugate **g**Q of Q, the infinite point of the line $Q\mathbf{g}Q$ and its isogonal conjugate (singular focus). Its equation is :

$$\sum_{\text{cyclic}} [p^2(c^2q - b^2r) + a^2qr(q-r)]\, x\,(c^2y^2 - b^2z^2) + 2\left(\sum_{\text{cyclic}} b^2c^2p^2(q-r)\right) xyz = 0.$$

$n\mathcal{K}_Q$ can be seen as the locus of the foci of the inconics centered on the line $Q\mathbf{g}Q$ hence it also contains the center and the foci of the inconic $\mathcal{IC}(\mathbf{g}Q)$.

2.4.3 Pencil of cubics passing through Q

In general, $p\mathcal{K}$ and $n\mathcal{K}_Q$ are two distinct isogonal circum-cubics passing through Q and **g**Q. They generate a pencil of cubics stable under isogonal conjugation and the nine common points of all the cubics are A, B, C, Q, **g**Q and the four foci of $\mathcal{IC}(Q)$.

This pencil contains, apart $p\mathcal{K}$ and $n\mathcal{K}_Q$, several remarkable cubics :

1. a central cubic $\mathcal{K}_c(Q)$ with center Q having equation
$$\sum_{\text{cyclic}} b^2c^2\, p\, x\,(ry - qz)[2p(y+z) - (q+r-p)x] = 0,$$
which contains :
– the infinite points of $p\mathcal{K}(K, \mathbf{g}Q)$,
– the reflection Z of **g**Q in Q,
– the common points of $p\mathcal{K}(K, Z)$ and the circumcircle.
This central cubic is a $\mathcal{K}_0$ (without term in xyz) if and only if Q lies on the Grebe cubic K102.
It becomes an isogonal focal central cubic when Q lies on the circumcircle (see CL001 in [2]).
2. the isogonal transform $\mathcal{K}_c^*(Q)$ of $\mathcal{K}_c(Q)$ which contains :
– Q, **g**Q, the tangent at **g**Q being the line $Q\mathbf{g}Q$,
– the common points of $p\mathcal{K}(K, \mathbf{g}Q)$ and the circumcircle,
– the infinite points of $p\mathcal{K}(K, Z)$,
– the isogonal conjugate **g**Z of Z.
3. an equilateral cubic $\mathcal{K}_{60}(Q)$ if and only if Q lies on the McCay cubic K003. See §2.4.1 above.
4. a cubic passing through the perspector P of the inconic with a rather complicated equation.

2.4.4 Another pencil of cubics

In a similar way, $p\mathcal{K}$ and $n\mathcal{K}_\infty$ are two distinct isogonal circum-cubics generating another pencil of cubics stable under isogonal conjugation. It contains several pairs of remarkable cubics as follows.

1. $\mathcal{K}_G(Q)$ and $\mathcal{K}_K(Q)$

   - $\mathcal{K}_G(Q)$ contains G, the common points of the circumcircle and the Thomson cubic, the infinite points of $p\mathcal{K}(K,S)$, where S is the homothetic of Q under $h(G,3)$.
     Its equation is :

$$\sum_{\text{cyclic}} x[(3p+q-r)c^2y^2-(3p-q+r)b^2z^2]+2\left(\sum_{\text{cyclic}}(b^2-c^2)p\right)xyz=0.$$

   - $\mathcal{K}_K(Q)$ contains K, the infinite points of the Thomson cubic and the common points of the circumcircle and $p\mathcal{K}(K,S)$, with S as above.
     Its equation is :

$$\sum_{\text{cyclic}} x[(3p-q+r)c^2y^2-(3p+q-r)b^2z^2]+2\left(\sum_{\text{cyclic}}(b^2-c^2)p\right)xyz=0.$$

2. $\mathcal{K}_O(Q)$ and $\mathcal{K}_H(Q)$

   - $\mathcal{K}_O(Q)$ contains O, the midpoints of ABC, $\mathbf{a}Q$, the infinite points of $p\mathcal{K}(K,\mathbf{aa}Q)$, the common points of the circumcircle and $p\mathcal{K}(K,\mathbf{a}Q)$. The tangents at A, B, C concur at $\mathbf{g}Q$.
     Its equation is :

$$\sum_{\text{cyclic}} a^2(q+r-p)(y-z)yz-2\left(\sum_{\text{cyclic}}(b^2-c^2)p\right)xyz=0.$$

   - $\mathcal{K}_H(Q)$ contains H, $\mathbf{ga}Q$, $\mathbf{aa}Q$, the points of tangency of $\mathcal{IC}(Q)$ with the sidelines of ABC, the infinite points of $p\mathcal{K}(K,\mathbf{a}Q)$, the common points of the circumcircle and $p\mathcal{K}(K,\mathbf{aa}Q)$.
     Its equation is :

$$\sum_{\text{cyclic}}(q+r-p)x^2(c^2y-b^2z)+2\left(\sum_{\text{cyclic}}(b^2-c^2)p\right)xyz=0.$$

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