Pairs and Triads of points on the Neuberg Cubic connected with Euler Lines and Brocard Axes Isometric Parallel Chords

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Abstract

The Neuberg cubic K001 has numerous properties but, in this paper, we will focus on those related with Euler lines and Brocard axes of certain triangles. This will characterize interesting pairs, triads and other groups of points and will yield to parallel isometric chords on this remarkable cubic .

1 Preliminaries

The Neuberg cubic K001 is the pivotal isogonal cubic with pivot the infinite point X_{30} of the Euler line (E) of the reference triangle ABC. Recall that it is a circular cubic with focus X_{110} and with real asymptote the parallel to (E) at the antipode X_{74} of X_{110} on the circumcircle (O) of ABC. It contains a good number of ETC centers and, in particular, the reflection X_{399} of X_3 about X_{110} which will have a great importance in the sequel.

Among its numerous properties (see [1] for instance), one is well-known and directly related to Euler lines of certain triangles, namely

Proposition 1 *The Euler lines (E_a), (E_b), (E_c) of triangles PBC, PCA, PAB concur (at M on (E)) if and only if P lies on K001 (together with (O) and the line at infinity which we do not consider in the sequel)*

If Euler lines are replaced by Brocard axes, we have an analogous property which is

Proposition 2 *The Brocard axes (B_a), (B_b), (B_c) of triangles PBC, PCA, PAB concur (at M on the Brocard axis (B) of ABC) if and only if P lies on K001 (together with (O) and the line at infinity which we do not consider in the sequel)*

We wish to characterize the correspondence between P on K001 and M on (E) or on (B).

We begin with several lemmas, some well-known and documented, we shall need in this paper.

Lemma 1 (definition of X_{399}) *If L_A, L_B, L_C are the parallels at A, B, C to (E) then their reflections L'_A, L'_B, L'_C in the sidelines BC, CA, AB concur at the Parry reflection point X_{399}. See [7].*

Recall that X_{399} lies on K001 and that it is the reflection of $O = X_3$ in X_{110}.

Lemma 2 *The lines (E_a) and (E) are parallel if and only if P lies on the circum-conic whose isogonal transform is the line passing through X_{399} and the $A-$vertex of the cevian triangle of X_{323}.*

Note that X_{323} is the cevapoint of X_6 and X_{399}.

Lemma 3 (definition of X_{1138}) *The lines (E), (E_a), (E_b), (E_c) are parallel if and only if P is the isogonal conjugate X_{1138} of X_{399}.*

In other words, if $P = X_{1138}$ then $M = X_{30}$.

Lemma 4 *For any P on* K001*, the line PX_{399} meets the cubic again at a third point Q which also lies on the line passing through the orthocenter $H = X_4$ and X_{30}/P^*.*

Here, P^* is the isogonal conjugate of P (hence on the cubic) and X_{30}/P^* is the Ceva conjugate of X_{30} and P^* (also on the cubic). The lemma is obvious since two isogonal points on K001 must be collinear with the pivot X_{30} and two $X_{30}-$Ceva conjugate points must be collinear with the isopivot X_{74}.

2 Neuberg cubic and Euler lines

The correspondence (P on K001) $\mapsto$ (M on (E)) is trivial after proposition 1 and it is clear that one chosen point on K001 will give one and only one point M on (E).

We wish now to investigate the correspondence in the "reverse" direction so let us take M on (E).

Proposition 3 *The Euler line (E_a) of triangle PBC contains M if and only if P lies on the cubic (K_a) described below.*

The two corresponding cubics (K_b), (K_c) are defined likewise.

The result is given by a fairly easy computation. The cubic (K_a) is a circular circum-cubic with focus M, with real asymptote the perpendicular at M to BC. It meets (O) at a sixth point O_a on the line L_A and it contains the vertices A_e, A_i of the equilateral triangles constructed externally, internally on BC (which is obvious since the Euler lines are then undetermined). Note that (K_a) has three concurring asymptotes passing through M hence the polar conic of M must split into the line at infinity and another line.

Since (K_a) has already seven common points with K001 (A, B, C, A_e, A_i, circular points J_1, J_2 at infinity), these two cubics must meet at two other points, say P, Q. See figure 1.

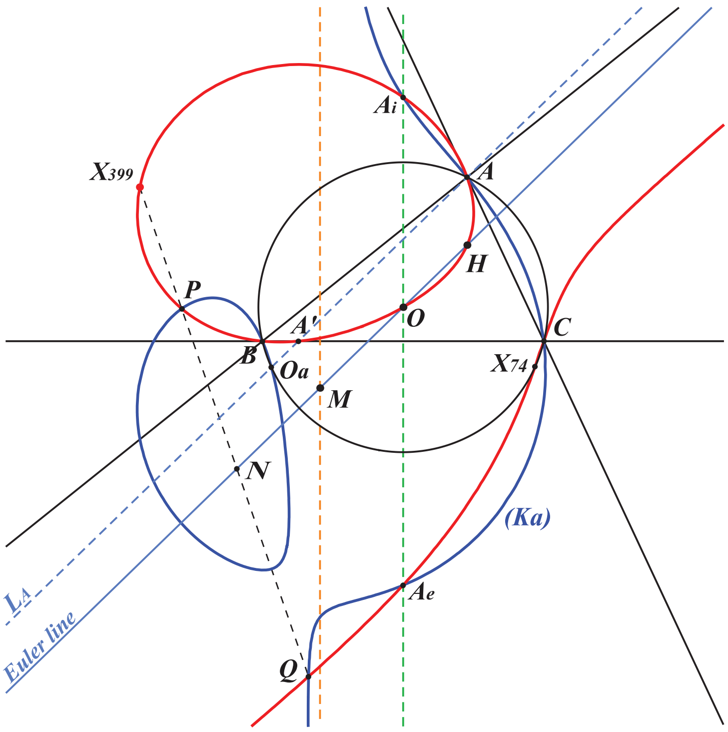

Figure 1: K001 and (K_a)

Remark : we eliminate the case $M = O$ since (K_a) would split into the perpendicular bisector of BC and (O).

Proposition 4 *For given M on (E), the three cubics (K_a), (K_b), (K_c) belong to a same pencil of circular circum-cubics with singular focus M and asymptotes concurring at M.*

This easily derives from the fact that the sum of the three cubics identically vanishes for every point M on the Euler line (E).

Proposition 5 *For given M on (E), the three cubics (K_a), (K_b), (K_c) meet again at two points P, Q which are the common points of the line $(L) = NX_{399}$ and the rectangular circum-hyperbola (H) passing through M, where N is the reflection of O in M. See figure 2.*

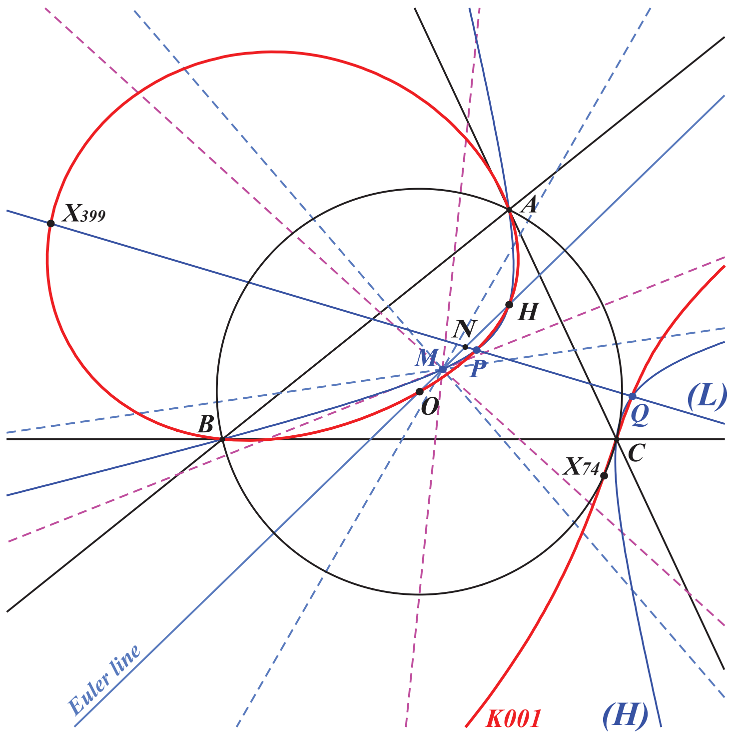

Figure 2: Construction of P, Q

The propositions above yield the following

Theorem 1 *For given finite M on (E), there are exactly two points P, Q on* K001 *such that the Euler lines of the six triangles PBC, PCA, PAB, QBC, QCA, QAB concur at M.*

Hence, $(M \text{ on } (E)) \mapsto ((P, Q) \text{ on K001})$ is a $(1, 2)$ correspondence.

Remarks

1. the knowledge of one of these two points P, Q on K001 gives the other by lemma 4.
2. the use of (L) and (H) gives an easy construction of K001.
3. X_{399} is the coresidual of A, B, C, H in K001.
4. if $M = X_{30}$ we have $N = P = X_{30}$ and then $Q = X_{1138}$ (the isogonal conjugate of X_{399}). Hence, the Euler lines of $X_{1138}BC$, $X_{1138}CA$, $X_{1138}AB$ are parallel to (E).

Examples

- with $M = X_2$ so $N = X_{381}$ the two points are X_{13}, X_{14}.
- with $M = X_5$ so $N = X_4$ the two points are X_4, X_{1263}.

3 Neuberg cubic and Brocard axes

The configuration is very similar to that with Euler lines and the correspondence (P on K001) $\mapsto$ (M on (B)) is trivial after proposition 2. It is clear that one chosen point on K001 will give one and only one point M on (B).

We also wish to investigate the correspondence in the "reverse" direction so let us take M on (B).

Proposition 6 *The Brocard axis (B_a) of triangle PBC contains M if and only if P lies on the bicircular circum-quartic (Q_a) described below.*

The two corresponding cubics (Q_b), (Q_c) are defined likewise.
Remark : when $M = O$, each quartic degenerates into the line at infinity, (O) and a perpendicular bisector of ABC which is not considered in the sequel.

(Q_a) meets the circum-circle at an eight point O_a lying on the line trough A and the intercept A' of the Brocard axis and the sideline BC. It also contains the vertices A_e, A_i mentioned above. See figure 3.

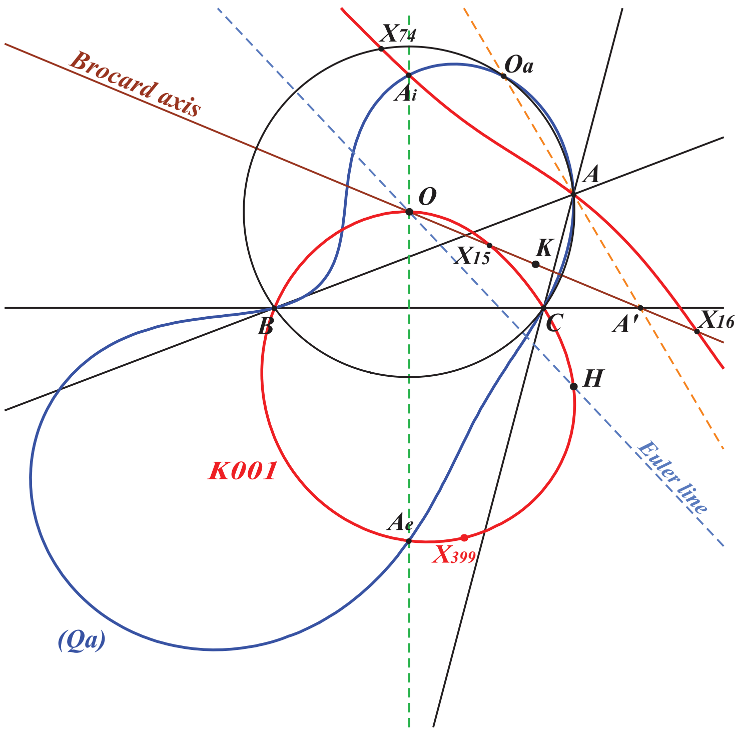

Figure 3: K001 and (Q_a)

Proposition 7 *The sum of these three quartics is the union of the line at infinity and K001.*

It follows that each quartic meets K001 at nine fixed points (namely A, B, C, each circular point at infinity counted twice, two vertices such as A_e, A_i) hence these must have three other common points say P_i, $i = 1, 2, 3$, and one of them is always real. Indeed, for each point P_i on K001 and (Q_a), the Brocard axes of triangles P_iBC, P_iCA, P_iAB contain M hence P_i must lie on (Q_b), (Q_c) as well. This gives

Proposition 8 *For M on (B), there are three points P_1, P_2, P_3 on K001 such that the Brocard axes of triangles P_iBC, P_iCA, P_iAB concur at M.*

After some quite heavy computations, we obtain that these three points lie on a same circle C_M that also contains X_{110} and X_{399} and whose radical axis with the circum-circle is the line $X_{110}M$. This gives

Theorem 2 *Any circle (γ) passing through X_{110} and X_{399} meets K001 at X_{399} and three other finite points P_1, P_2, P_3 such that the nine Brocard axes of triangles P_iBC, P_iCA, P_iAB concur at a certain point M on the Brocard axis (B) of ABC. Moreover, M is the intersection of (B) and the radical axis of (γ) and (O), meeting (O) again at N.*

Hence, $(M \text{ on } (B)) \mapsto (P_1, P_2, P_3 \text{ on K001})$ is a $(1, 3)$ correspondence. See figure 4.

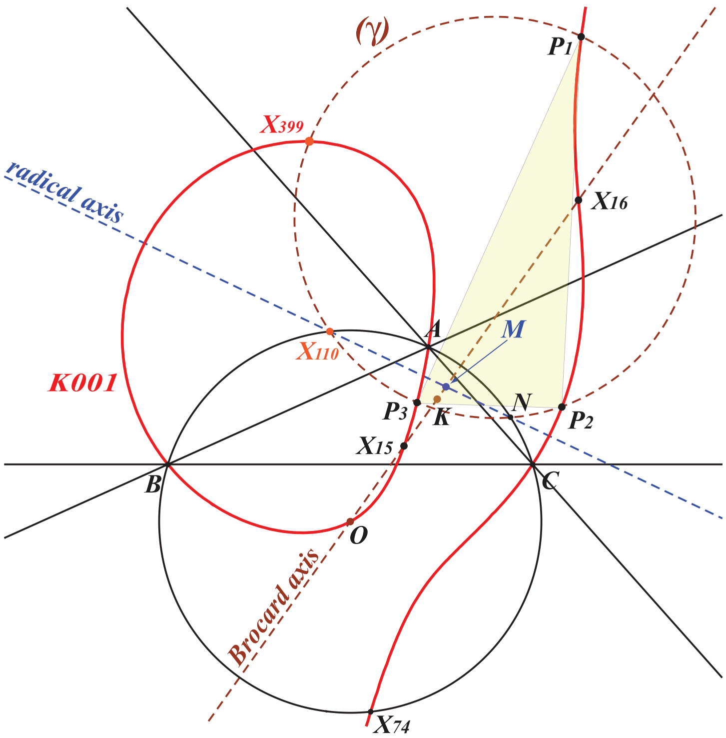

Figure 4: K001 and the circle (γ)

Examples

- with $M = X_{61}$, the three points are X_{14}, X_{15}, X_{1337} and $N = X_{2380}$.
- with $M = X_{62}$, the three points are X_{13}, X_{16}, X_{1338} and $N = X_{2381}$.

A conic construction of P_1, P_2, P_3

Let M be a point on the Brocard axis and let N be the second intersection of the line $(L_2) = MX_{399}$ with the circumcircle of ABC.

The circumcircle C_M of N, X_{110}, X_{399} is centered at S on the perpendicular bisector (Δ) of X_{110}, X_{399}. Let S' be the reflection of S about the intersection of (Δ) and the Euler line.

The line $(L_1) = S'X_{30}$ and its isogonal transform (H_1) meet at two points E_1, E_2 on K001.

(L_2) and (H_1) meet at two points E_3, E_4 and then the rectangular hyperbola (H) passing through X_{399}, E_1, E_2, E_3, E_4 meets C_M at X_{399} and the three requested points P_1, P_2, P_3. See figure 5.

Note that this construction is valid when all the points E_1, E_2, E_3, E_4 are real.

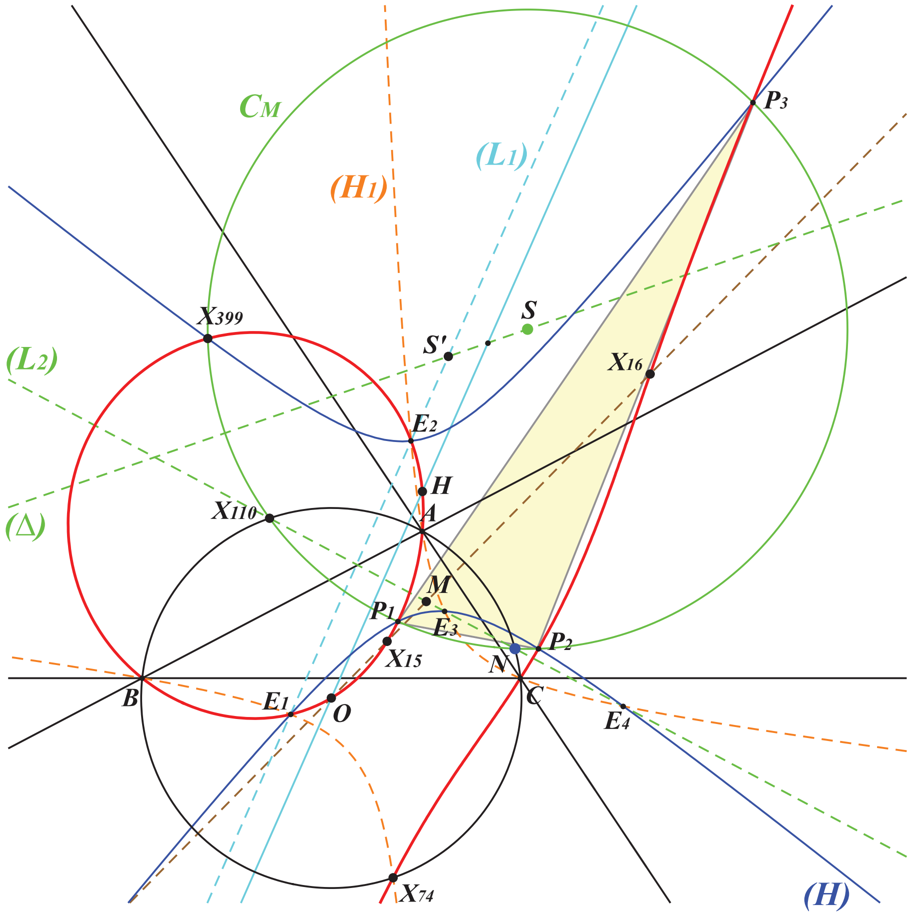

Figure 5: Construction of P_1, P_2, P_3

4 Another configuration related with Euler lines

4.1 Generalities

There is another related configuration although the locus of common points is not a line but the complicated sextic Q093, namely

Proposition 9 *Let $P_aP_bP_c$ be the pedal triangle of P. The Euler lines (E_a), (E_b), (E_c) of triangles PP_bP_c, PP_cP_a, PP_aP_b concur if and only if P lies on K001 (together with the line at infinity).*

Now, given a fixed point $Q = u : v : w$, we seek P such that the Euler line (E_a) of PP_bP_c contains Q. We find a rectangular hyperbola (C_a) whose asymptotes are parallel to the bisectors of A in ABC. Moreover (C_a) contains A, the reflection A_1 of A in Q (for which the circumcenter of PP_bP_c is Q) and another point A_2 (for which the orthocenter of PP_bP_c is Q) that can be constructed as follows.

If A_2' (resp. A_2'') is the intersection of the parallel at Q to AB (resp. AC) with the sideline AC (resp. AB) then the perpendiculars at A_2' to AC and at A_2'' to AB meet at the requested point A_2.

Since we know five points on (C_a), its construction can be realized. Note that its center O_a lies on the parallel at Q to the line AQ^* where Q^* is the isogonal conjugate of Q. The reflection A' of A in O_a lies on the line QA_2 and on the parallel at A_1 to the line AQ^*. See figure 6.

Thus, for any point P on (C_a), the Euler line (E_a) of PP_bP_c contains Q. Furthermore, the circumcenter $O_a(P)$ and the orthocenter $H_a(P)$ of PP_bP_c lie on two rectangular hyperbolas having their asymptotes parallel to those of (C_a) and passing through A, Q. The former also contains O_a and the latter also contains A'. See figure 7.

Two other rectangular hyperbolas (C_b), (C_c) with centers O_b, O_c are defined likewise and all three generate a net of conics whose Jacobian is in general a focal cubic $J(Q)$ whose orthic line is the line

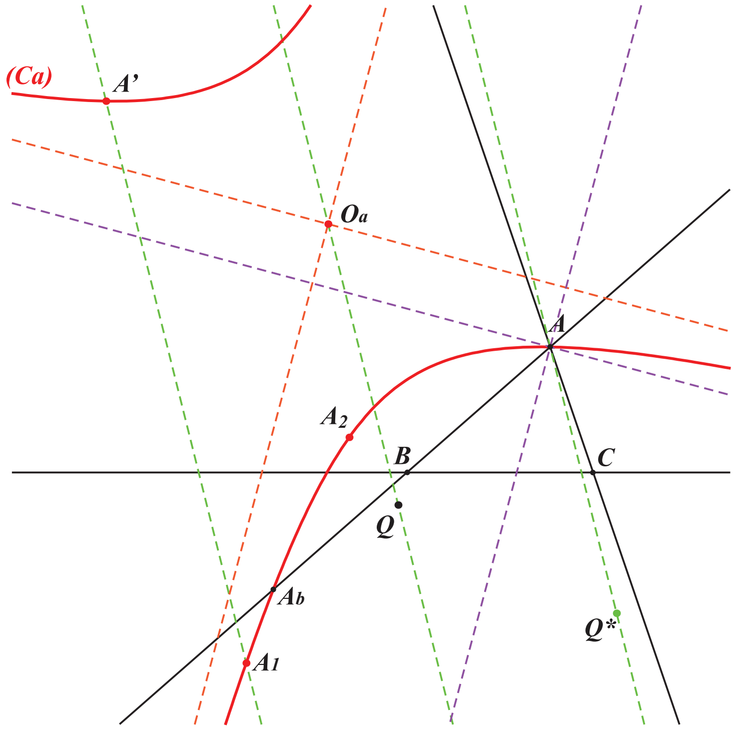

Figure 6: The rectangular hyperbola (C_a)

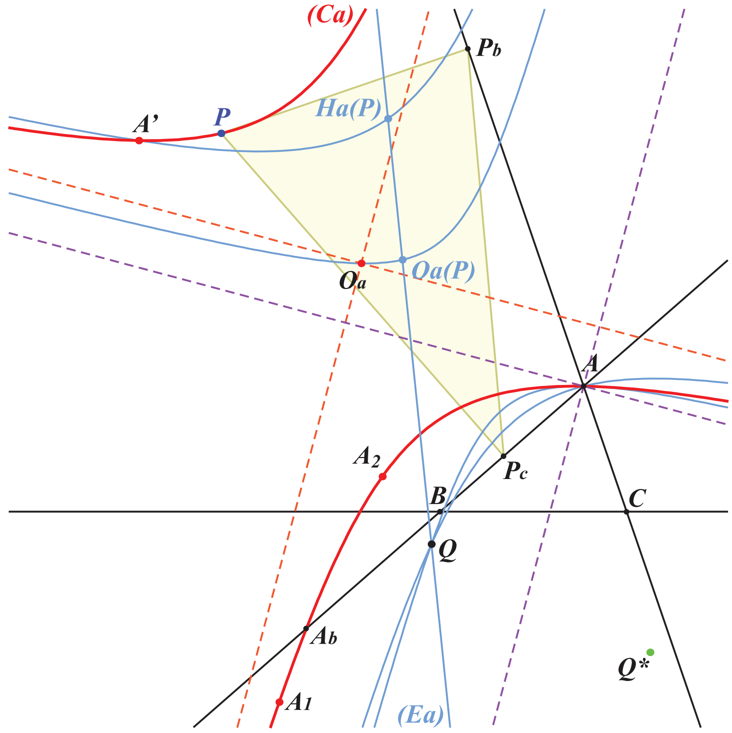

Figure 7: The rectangular hyperbola (C_a) and the Euler line (E_a)

HQ, hence the polar conics of H and Q (and consequently of any point on HQ) are also rectangular hyperbolas. Note that the infinite point of the line HQ is the third infinite point of $J(Q)$.

When $Q = H$, the three rectangular hyperbolas are in a same pencil hence the Jacobian identically vanishes, see §4.3 below.

Remarks

1. When Q lies on the Napoleon cubic K005, the tangents at A, B, C to (C_a), (C_b), (C_c) concur (on K001) and the triangles ABC, $O_aO_bO_c$ are perspective (at a point also on K001).

2. In general, Q is not on $J(Q)$ unless it is a point of the McCay cubic K003.

3. The singular focus F of $J(Q)$ lies on the line HQ if and only if Q lies on the parallel at H to one of the asymptotes of K003. In such case, $J(Q)$ is a central focal cubic.

Now, according to the number N of common points P_i, $(0 \leqslant i \leqslant 4)$ of the three rectangular hyperbolas, the nature of $J(Q)$ can be precised :

N	nature of $J(Q)$
0	an elliptic (non-unicursal) focal cubic
1	a nodal (unicursal) focal cubic whose node is P
2	a line and a circle meeting at these two points
3	three lines through the three points
4	identically vanishes

Let then P be one common point of (C_a), (C_b), (C_c). When we express that $P = p : q : r$ lies on the three hyperbolas, we obtain three conditions which are linear in u, v, w hence, in general, these conditions are not simultaneously realized but, by eliminating the coordinates of Q, we find that P must lie on K001 which is, unsurprisingly, proposition 9.

These conditions are of second degree in p, q, r and the elimination of these gives the locus of Q which is the circular sextic Q093. Hence, for any point Q on Q093, the three rectangular hyperbolas have at least one common point P (on K001) such that the Euler lines (E_a), (E_b), (E_c) concur at Q. See figure 8.

Properties of Q093

Q093 contains H (quadruple), X_1 and the excenters, X_{30} and the infinite points of the Napoleon cubic K005, X_{125}, X_{140}, X_{3574}, the vertices of the orthic triangle, the $O-$Ceva conjugates of the infinite points of the McCay cubic K003 which are double points on the curve.

According to the position and the nature of Q with respect to Q093 and therefore the number of common points of the three rectangular hyperbolas, we can revisit the nature of $J(Q)$ with some examples given in the next paragraph.

4.2 Some examples of Jacobian

Let Q be a point in the plane.

4.2.1 Q is not on Q093

The three rectangular hyperbolas have no common point and $J(Q)$ is a focal elliptic cubic.

An interesting example is $J(X_3)$ since the cubic contains X_3, X_{20}, X_{30}, X_{74} which is the singular focus whose polar conic is the circle also passing through O and X_{20}. The orthic line is the Euler line and the real asymptote is its parallel through X_{110}. See figure 9.

4.2.2 Q is a simple point on Q093

The three rectangular hyperbolas have one common point P on K001 and $J(Q)$ is a strophoid with node P.

One case is special and is obtained when $Q = X_1$ (or an excenter). Indeed, each rectangular hyperbola splits into one internal bisector of ABC and a perpendicular hence $P = X_1$ and $J(X_1)$ also

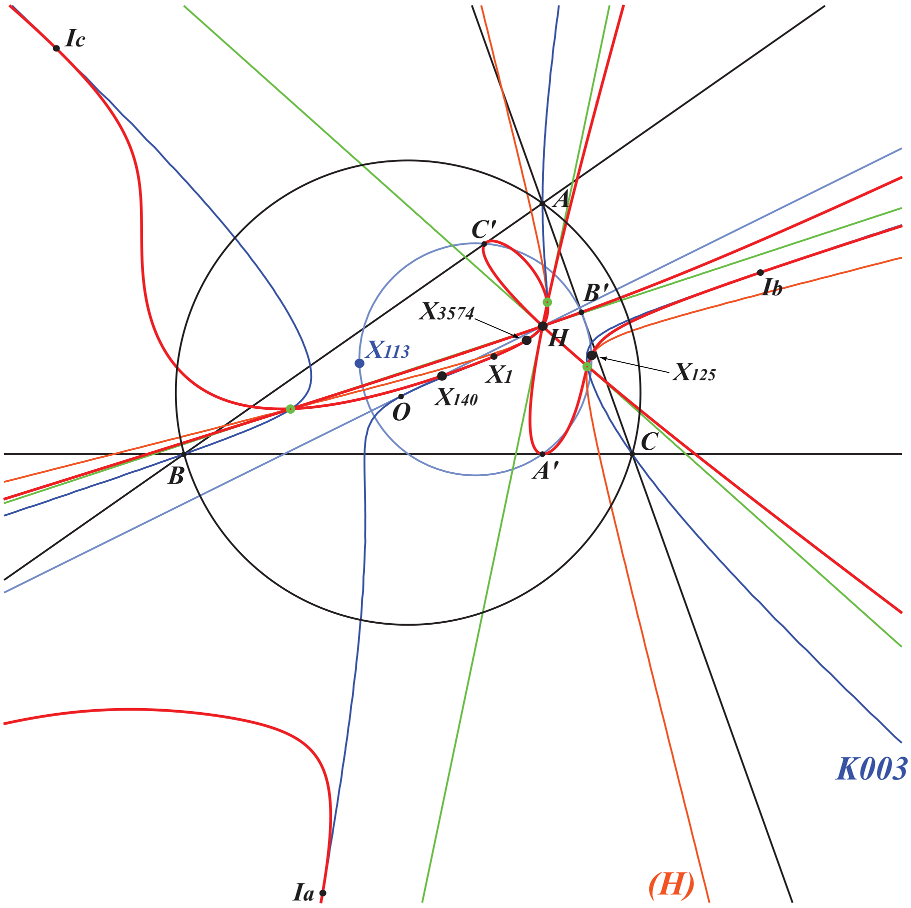

Figure 8: The circular sextic Q093

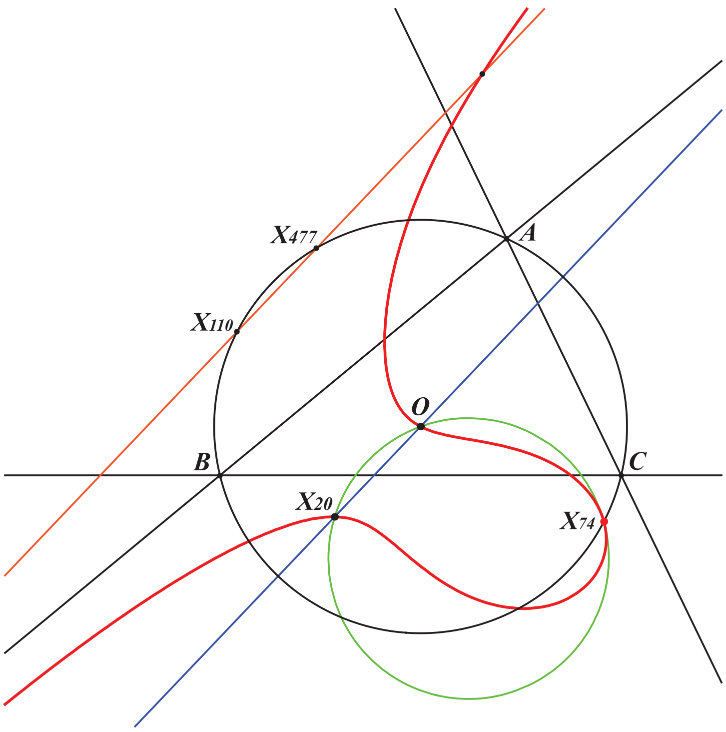

Figure 9: The Jacobian $J(X_3)$

contains the three centers of these decomposed hyperbolas. $J(X_1)$ is a strophoid with node X_1 and the relative Euler lines (E_a), (E_b), (E_c) are the bisectors. The singular focus is the intersection of the lines X_1X_{1361} and X_3X_{214}. See figure 10.

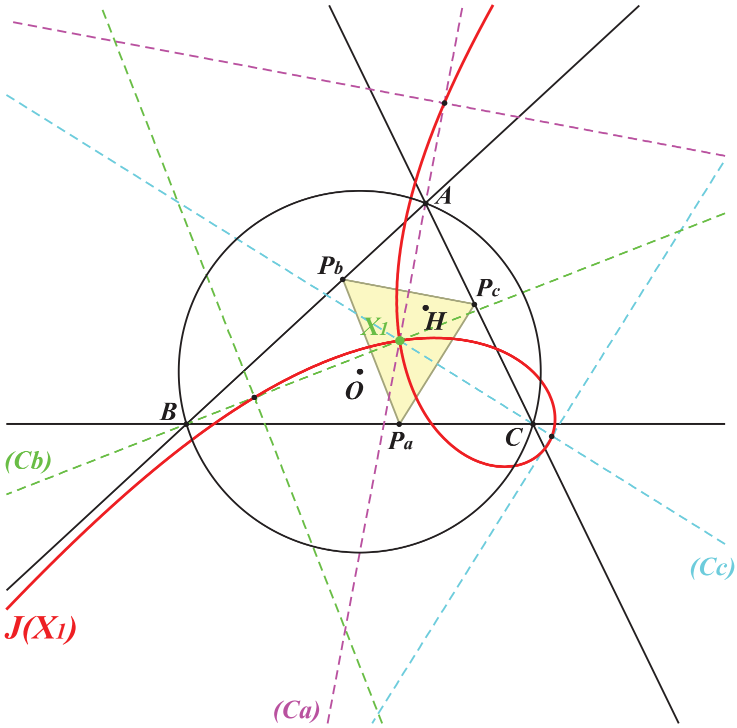

Figure 10: The Jacobian $J(X_1)$

4.2.3 Q is a double point on Q093

The three rectangular hyperbolas have two common points P_1, P_2 on K001 and $J(Q)$ splits into a line and a circle. There are three double points on Q093 which are represented in green on figures 8 and 11. They are $O-$Ceva conjugates of the infinite points of the McCay cubic K003 hence they lie on the parallels at H to the asymptotes of K003. They are the common points (apart X_{125}) of the bicevian conic $C(G, O)$ (the $O-$Ceva conjugate of the line at infinity) and the image of the Jerabek hyperbola (which is the polar conic of H in K003) under the homothety with center H, ratio 1/2.

It follows that, if Q is one of these three points, $J(Q)$ splits into the line HQ (which is parallel to one of the asymptotes of K003) and the circle whose diameter is P_1P_2 where P_1, P_2 are the intersections of HQ and K001. See figure 11.

Consequently, there are six Euler lines passing through each of these three points Q. See figure 12.

4.2.4 $Q \neq H$ cannot be a triple point on Q093

Indeed, the line HQ would meet Q093 at seven points and this cannot occur since Q093 is a sextic.

4.3 The special case $Q = H$

Recall that, when $Q = H$, the Jacobian $J(H)$ of the three rectangular hyperbolas (C_a), (C_b), (C_c) identically vanishes hence these belong to a same pencil and therefore have four common points P_i, $i \in \{1, 2, 3, 4\}$ forming an orthocentric system. It follows that, for any given point which is not one of the P_i, there is one and only one rectangular hyperbola of the pencil passing through this point.

One of the most remarkable of these hyperbolas is H_{30} passing through the infinite point X_{30} of the Euler line and having a common asymptote with K001, namely the line $X_{30}X_{74}$. See figure 13.

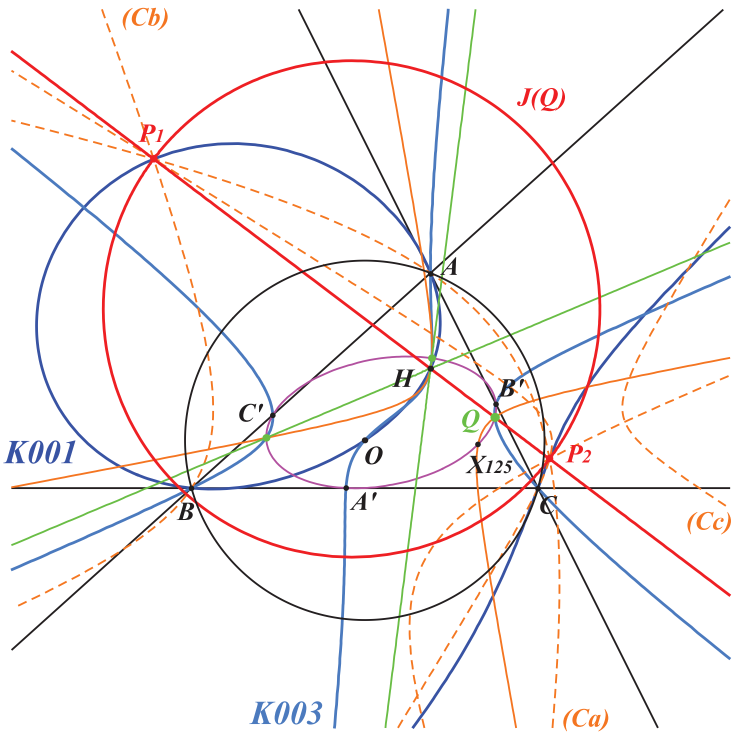

Figure 11: Decomposed Jacobian

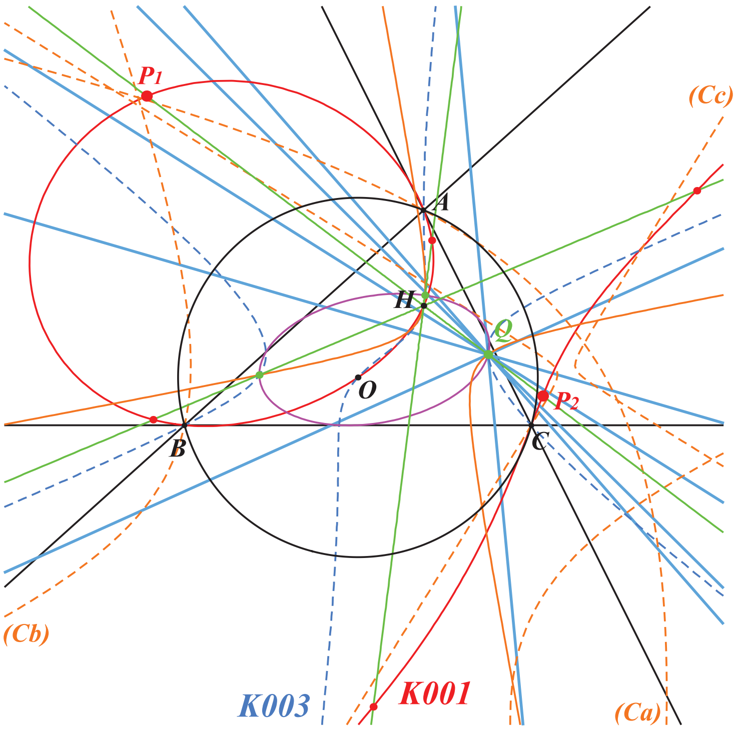

Figure 12: Six Euler lines through Q

H_{30} is actually obtained from the polar conic of X_{30} in K001 under the translation with vector $\overrightarrow{OH}$. This polar conic is a diagonal rectangular hyperbola $\mathcal{H}$ passing through the in/excenters and also X_5, X_{30}, X_{395}, X_{396}, X_{523}, X_{1749}.

Now, since $\mathcal{H}$ bisects any segment PP^* of isogonal conjugate points on K001, we have $\overrightarrow{P_iP_i^*} = -2\,\overrightarrow{OH} = \overrightarrow{HX_{20}}$. This gives

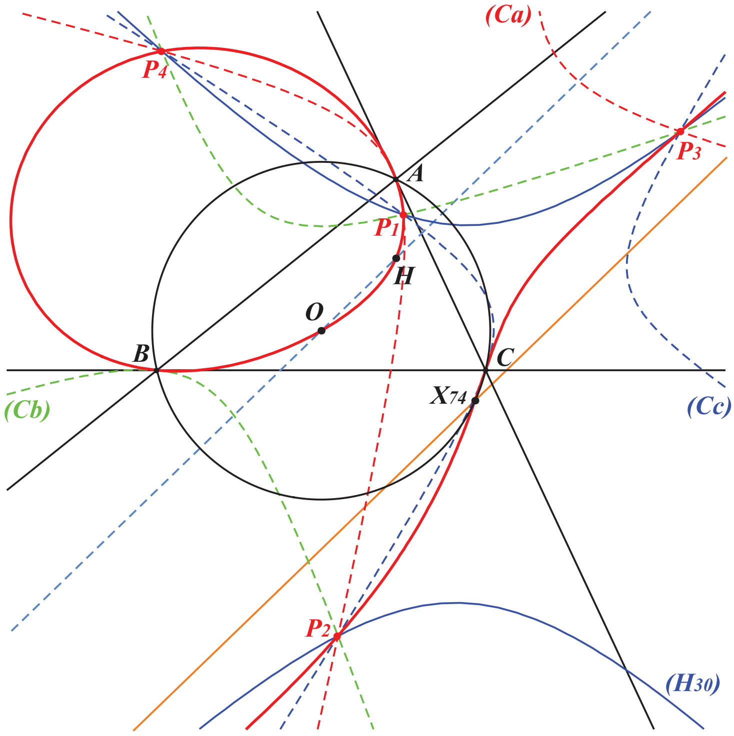

Figure 13: The pencil of rectangular hyperbolas (C_a), (C_b), (C_c) when $Q = H$

Theorem 3 *There are four points P on* K001 *such that the twelve Euler lines of triangles PP_bP_c, PP_cP_a, PP_aP_b concur at H. Furthermore, these points are the only points P in the plane such that* $\overrightarrow{PP^*} = -2\,\overrightarrow{OH} = \overrightarrow{HX_{20}}$.

The figure 14 shows one of these points P with its pedal triangle $P_aP_bP_c$ and the three corresponding Euler lines through H. The isogonal conjugates of P_i are labelled Q_i and the four parallel chords P_iQ_i with length $2\,OH$ are also represented. This will be generalized in the next paragraph. See figure 14.

Remark : the midpoint of any pair of points $P_i\,Q_j$, $i \neq j$, lies on the circumcircle and the midpoint of any pair of points $P_i\,Q_i$ lies on $\mathcal{H}$.

5 Isometric parallel chords on the Neuberg cubic

Let k be a real number. We wish to characterize the points P such that

$$\overrightarrow{PP^*} = k\,\overrightarrow{OH} = \overrightarrow{OQ}, \tag{1}$$

where P^* is the isogonal conjugate of P. This will generalize Theorem 3 above.

One trivial case is obtained when $k = 0$ since P must be one in/excenter of triangle ABC, these four points obviously lying on K001 and on the diagonal rectangular hyperbola $\mathcal{H}$ above.

If $k \neq 0$, P (and P^*) must lie on K001 since the line PP^* is parallel to the Euler line.

5.1 The general case

The condition (1) on P shows that P must be a common point of three rectangular hyperbolas (H_a), (H_b), (H_c) belonging to a same pencil $\mathcal{F}_k$ that also contains the image $\mathcal{H}_k$ of $\mathcal{H}$ under the translation with vector $\overrightarrow{T} = -1/2\,\overrightarrow{OQ} = -k\,\overrightarrow{ON}$, where $N = X_5$ is the nine point center of ABC. See figure 15.

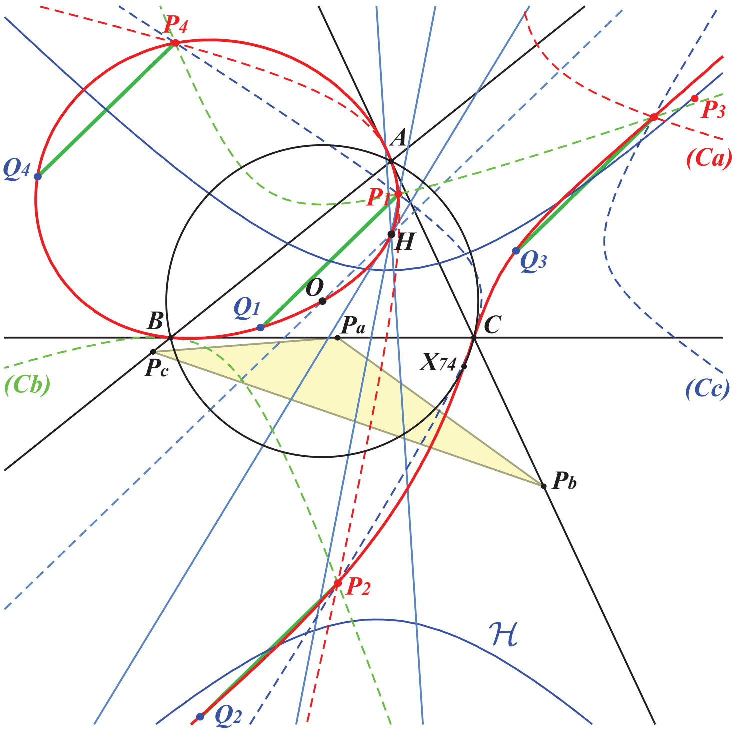

Figure 14: Euler lines when $Q = H$

Properties of (H_a)

1. It contains A and the tangent passes through X_{74}.
2. It has two asymptotes parallel to the bisectors at A in the triangle ABC.
3. It contains A' defined by $\overrightarrow{AA'} = \overrightarrow{QO}$.
4. Its center O_a is the midpoint of AA' hence on the $A-$cevian line of X_{30}.
5. It meets the lines AO, AH again at A_O, A_H which are on the parallel at O_a to the $A-$cevian line of X_5.
6. It belongs to the pencil of rectangular hyperbolas generated by the bisectors at A and the union of the line at infinity with AX_{74}.

5.2 Properties of $\mathcal{F}_k$

Let k be a given real number.

1. The rectangular hyperbolas of $\mathcal{F}_k$ have their centers on the circle $\mathcal{C}_k$ with radius R (that of the circumcircle (O) of ABC) and center Ω_k such that $\overrightarrow{O\Omega_k} = -k/2\,\overrightarrow{OH} = -1/2\,\overrightarrow{OQ}$.
2. If P_i, $i \in \{1, 2, 3, 4\}$, are the four basis points of $\mathcal{F}_k$ then the midpoints of P_iP_j lie on $\mathcal{C}_k$ and the midpoints of $P_i^*P_j^*$ lie on the image of $\mathcal{C}_k$ under the translation with vector $\overrightarrow{OQ}$.
3. The points P_i^* are therefore the basis points of the pencil $\mathcal{F}_{-k}$.
4. The lines $P_iP_j^*$ and $P_i^*P_j$ meet on (O).

Figure 16 shows $\mathcal{F}_k$ when $k = -5/2$ in which the P_i^* are labelled Q_i. In this case, one can find four isometric parallel chords to K001 which is obviously not always true.

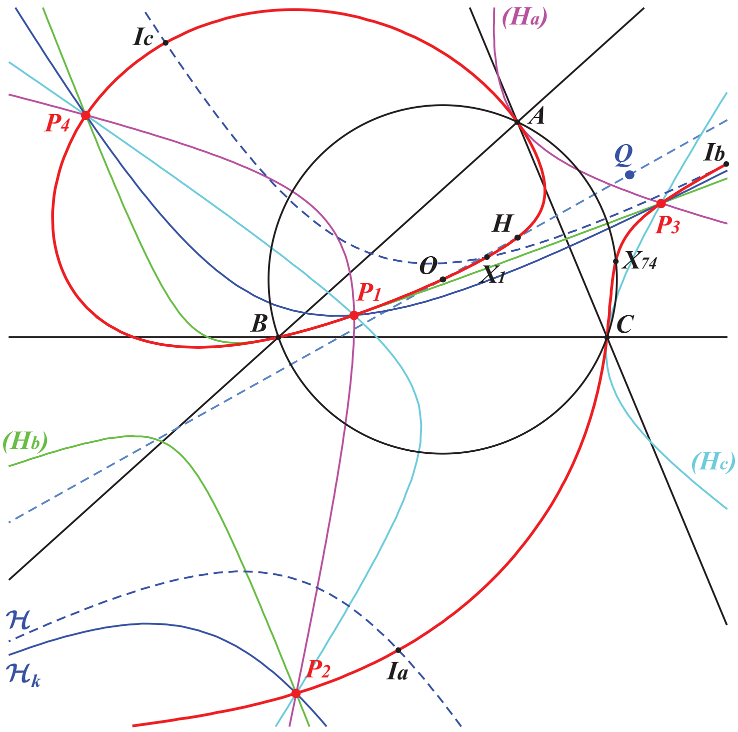

Figure 15: The rectangular hyperbolas (H_a), (H_b), (H_c) with $\mathcal{H}_k$ and $\mathcal{H}$

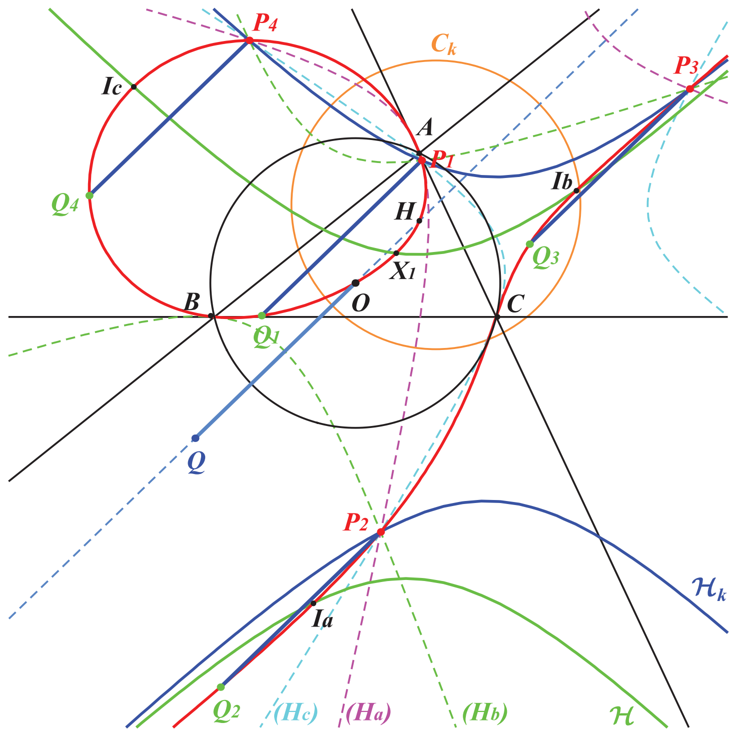

Figure 16: K001 with four isometric parallel chords

5.3 Special cases

1. Recall that the basis points of $\mathcal{F}_0$ are the in/excenters of ABC.
2. The pencil $\mathcal{F}_{-2}$ is the one we met in section 4.3.

3. The basis points of $\mathcal{F}_1$ are O and three other points U_a, U_b, U_c which are connected with pivotal equilateral isocubics.

   Indeed, there is in general one and only one pivotal equilateral cubic with given pivot but there are infinitely many such cubics when the pivot is one of the points U_a, U_b, U_c. See [4], §6.5.

5.4 Number of isometric parallel chords

Recall that the in/excenters are the only finite points where the tangents are parallel to the Euler line (and to the real asymptote). Since K001 is formed by an oval and an infinite inflexional branch, there must be two in/excenters on the oval and two in/excenters on the infinite branch which can contain two chords of any possible length.

Hence, the number of isometric parallel chords depends of the number of such chords on the oval which can be 0, 1 or 2 and therefore 2, 3 or 4 on the cubic.

It is in particular possible to have only three such chords (one only on the oval) when (H_a), (H_b), (H_c) are tangent at one of the points P_i. See figure 17.

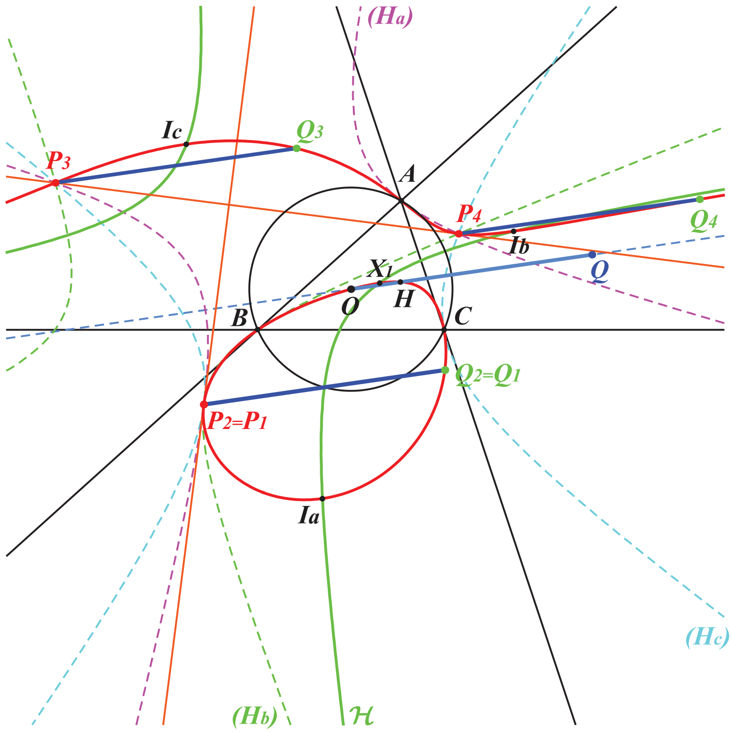

Figure 17: K001 with only three isometric parallel chords

The computation of the length of this longest chord on the oval is difficult and requires the resolution of an equation of degree four. See [2] for further developments and related cubics.

References

[1] Gibert B., *Catalogue of Triangle Cubics*, available at
http://pagesperso-orange.fr/bernard.gibert/

[2] Gibert B., *Isometric Parallel Chords on the Neuberg Cubic*, available at
http://pagesperso-orange.fr/bernard.gibert/

[3] Gibert B., *Pseudo-Pivotal Cubics and Poristic Triangles*, available at
http://pagesperso-orange.fr/bernard.gibert/

[4] Gibert B., *Special Isocubics in the Triangle Plane*, available at
http://pagesperso-orange.fr/bernard.gibert/

[5] Kimberling C., *Triangle Centers and Central Triangles*, Congressus Numerantium 129, Winnipeg, Canada, 1998.

[6] Kimberling C., *Encyclopedia of Triangle Centers*,
http://faculty.evansville.edu/ck6/encyclopedia/ETC.html

[7] Parry C., *Problem 10637*, American Mathematical Monthly 105 (1998) 68.