Another kind of Lemoine cubics

Bernard Gibert

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Abstract

We revisit the configuration seen in our Lemoine cubics paper [3] and study another kind of related nodal cubics.

1 Introduction

Let $P = p : q : r$ be a fixed finite point not lying on a sideline of the reference triangle ABC nor on its circumcircle $\mathcal{O}$. Let $P_aP_bP_c$ be the pedal triangle of P which is therefore a proper triangle.

If M is a variable point, let $M_a = AM \cap PP_a$, $M_b = BM \cap PP_b$ and $M_c = CM \cap PP_c$.

These three latter points are collinear if and only if M lies on the generalized Lemoine cubic $\mathcal{K}(P)$ which is studied in [3].

In this paper, we consider the locus of M such that these same three points all lie on a same circum-conic or, equivalently, their isogonal (or isotomic or " isoanything") conjugates are collinear.

It is obvious that when P lies on an altitude of ABC at least one of these points coincide with one vertex of ABC in which case M can be any point of the plane. This is excluded in the sequel.

2 Definition and properties

If P is not one of the points excluded above then

Proposition 1 *The locus of M such that the points M_a, M_b, M_c all lie on a same circum-conic is a cubic here denoted by $\mathcal{L}(P)$.*

$\mathcal{L}(P)$ has the following properties.

1. $\mathcal{L}(P)$ is a nodal cubic with node P and nodal tangents the parallels at P to the asymptotes of the rectangular circum-hyperbola passing through P.

2. Apart P, $\mathcal{L}(P)$ contains the midpoint Q of HP, the points P_a, P_b, P_c and also the traces Q_a, Q_b, Q_c of the cevian lines of P in the pedal triangle $P_aP_bP_c$ of P. In other words, $\mathcal{L}(P)$ is a circum-cubic of $P_aP_bP_c$ and $Q_a = AP \cap P_bP_c$, etc.

3. The asymptotes of $\mathcal{L}(P)$ are parallel to those of the isogonal pivotal cubic $p\mathcal{K}(K, P')$ where P' is the reflection of P about O. In particular, $\mathcal{L}(P)$ is an equilateral cubic if and only if $P = O$ and then $\mathcal{L}(O)$ is the stelloid K258. See [1] for more details.

4. $\mathcal{L}(P)$ and $p\mathcal{K}(K, P')$ meet again at six finite points which lie on a same rectangular hyperbola $\mathcal{H}(P)$. $\mathcal{H}(P)$ contains P if and only if P lies on the McCay cubic K003. See §4.3.

5. These asymptotes are concurrent if and only if P lies on a circum-cubic passing through O, H, X_{83}. See §4.1.

6. If $\mathcal{L}(O)$ is not the stelloid K258, its orthic line is OP. This means that the polar conic of any point on OP is a rectangular hyperbola.

3 Construction and other properties

The construction of $\mathcal{L}(P)$ is an adaptation of that of $\mathcal{K}(P)$ given in [3], §5.

Let L_a be the line passing through $PQ_b \cap P_aP_b$ and $PQ_c \cap P_aP_c$ and call R_a its trilinear pole with respect to the triangle $P_aP_bP_c$ of P. Define L_b, R_b and L_c, R_c similarly. These points R_a, R_b, R_c are collinear on the line L.

The trilinear polar ℓ with respect to the triangle $P_aP_bP_c$ of any point on L meets the lines APQ_a, BPQ_b, CPQ_c at S_a, S_b, S_c respectively. Then, the lines P_aS_a, P_bS_b, P_cS_c concur at M on the cubic $\mathcal{L}(P)$.

Note that the line ℓ envelopes a conic $\mathcal{C}(P)$ and, since the tangents at P to $\mathcal{L}(P)$ are perpendicular, the point P must lie on the orthoptic circle of this conic.

This conic $\mathcal{C}(P)$ is then entirely defined by five of its tangents namely the sidelines of the pedal triangle of P and the nodal tangents at P to $\mathcal{L}(P)$. If S is the perspector of $\mathcal{C}(P)$ in the pedal triangle of P then the trilinear polar of S with respect to this same triangle is the line L defined above.

Note that $\mathcal{C}(P)$ is a diagonal conic in ABC if and only if P lies on the Darboux cubic K004.

It is therefore sufficient to take the line ℓ as a tangent to $\mathcal{C}(P)$ and proceed as above. See figure 1.

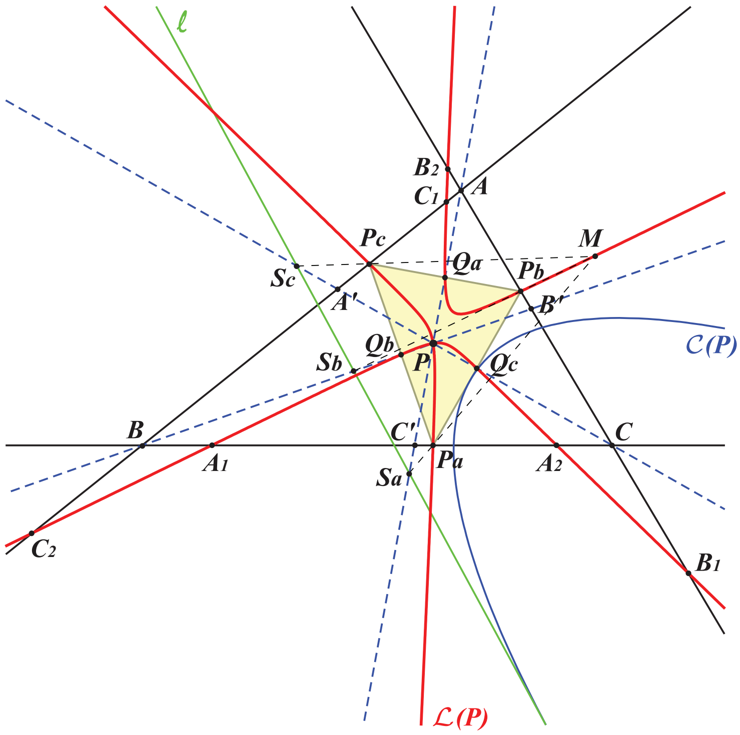

Figure 1: Construction of $\mathcal{L}(P)$

Consequence :

With suitably chosen tangents ℓ we find the other intersections of $\mathcal{L}(P)$ with the sidelines of ABC.

Let $A'B'C'$ be the cevian triangle of P. There are two tangents to $\mathcal{C}(P)$ drawn from A' and the construction above gives the two other points A_1, A_2 where $\mathcal{L}(P)$ intersects the sideline BC.

Note that the polar lines of A', B', C' in $\mathcal{C}(P)$ form a triangle inscribed in $P_aP_bP_c$ and perspective to ABC.

Remarks :

– the line L contains P if and only if P lies on the Kiepert hyperbola in which case L is parallel to the Euler line of ABC.

– this latter property also occurs when P lies on the circum-conic of ABC with center the Lemoine point K.

– the line L contains the midpoint Q of HP if and only if P lies on the cubic K663 = $p\mathcal{K}(H, G)$.

4 Special cases

4.1 $\mathcal{L}(P)$ with concurring asymptotes

This condition is realized if and only if P lies on a circum-cubic $\mathcal{K}_c$ = K664 passing through O, H, X_{83} and the isogonal conjugate X^*_{550} of X_{550}.

$\mathcal{K}_c$ is a member of the pencil generated by the McCay cubic K003, the union of the Euler line and $\mathcal{O}$ which also contains K009, K361, K405.

The isogonal transform $\mathcal{K}^*_c$ = K665 of $\mathcal{K}_c$ is a stelloid with radial center on the Euler line. $\mathcal{K}^*_c$ is a member of the pencil generated by the McCay cubic K003, the union of the line at infinity and the Jerabek hyperbola. This pencil also contains K026, K028, K080, K309, K358, K525, K581.

4.2 $\mathcal{L}(P)$ which is a $ps\mathcal{K}$ in $P_aP_bP_c$

This condition is realized if and only if P lies on the Darboux cubic K004. See [2] for definition and properties of $ps\mathcal{K}$ cubics.

In such case, the triangle $Q_aQ_bQ_c$ is perspective with $P_aP_bP_c$ or, equivalently, the tangents at P_a, P_b, P_c to $\mathcal{L}(P)$ are concurrent. Note that the center of of perspective and the point of concurrence of these tangents cannot lie on the cubic since it is a nodal cubic hence cannot be a pivotal cubic.

Recall that, in this case, $\mathcal{C}(P)$ is a diagonal conic in ABC.

4.3 $\mathcal{L}(P)$ such that $\mathcal{C}(P)$ is a parabola

This condition is realized if and only if P lies on the McCay cubic K003. Recall that, in this case, the rectangular hyperbola $\mathcal{H}(P)$ contains P. Note that the directrix of any such parabola must contain P.

5 Selected examples

5.1 $\mathcal{L}(O = X_3)$ = K258

This is undoubtedly the most interesting cubic since it is the only one that fulfills all three conditions mentioned above. It contains X_1, X_3, X_5, X_{39}, X_{2140}.

$P_aP_bP_c$ is the medial triangle $A'B'C'$ hence the complement of $\mathcal{L}(O)$ is a circum-cubic namely the third Musselman cubic K028 which is $ps\mathcal{K}(X_4, X_{264}, X_3)$. It follows that $\mathcal{L}(O)$ is a $ps\mathcal{K}$ with respect to the medial triangle with pseudo-pivot X_{216} and pseudo-isopivot X_{141}.

$\mathcal{L}(O)$ is a stelloid with three asymptotes parallel to those of the McCay cubic and concurring at X_{549}, the midpoint of GO. $\mathcal{C}(O)$ is the diagonal parabola with focus X_{122}, directrix the line through X_3, X_{64}, etc. $\mathcal{H}(O)$ is the Stammler hyperbola. The nodal tangents are parallel to the asymptotes of the Jerabek hyperbola. See figure 2.

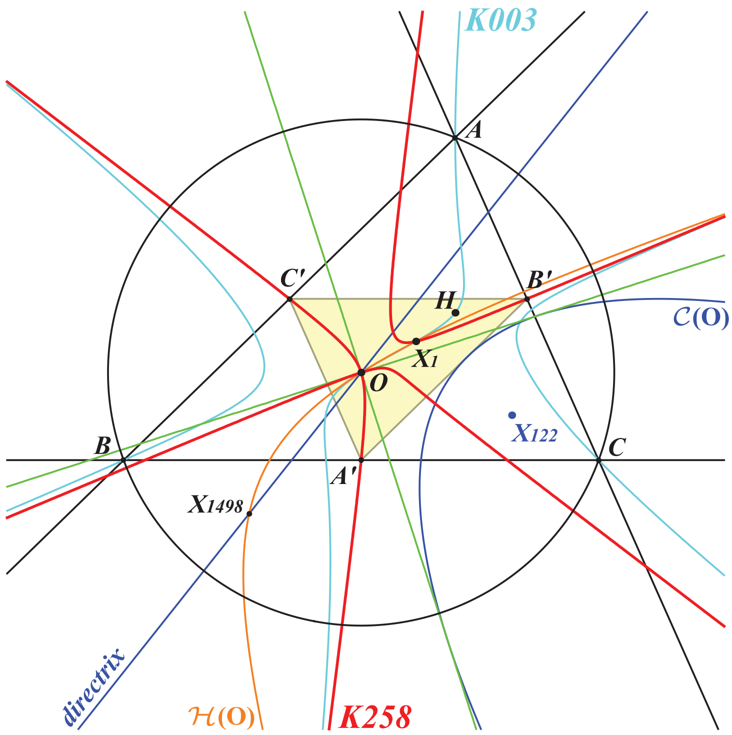

Figure 2: $\mathcal{L}(O)$ = K258

5.2 $\mathcal{L}(I = X_1)$ = K666

This cubic fulfills the last two conditions. It contains X_1, X_{65}, X_{946}, X_{2089}.

$P_aP_bP_c$ is the intouch triangle $A'B'C'$. The three asymptotes are parallel to those of K414 $= p\mathcal{K}(X_6, X_{40})$, the Orthocubic of the excentral triangle.

$\mathcal{L}(I)$ is a $ps\mathcal{K}$ with respect to the intouch triangle with pseudo-pivot X_{354} and pseudo-isopivot X_{226}.

$\mathcal{C}(I)$ is the diagonal parabola with focus X_{11}, directrix the line through X_1, X_3, etc.

The nodal tangents are parallel to the asymptotes of the Feuerbach hyperbola. See figure 3.

5.3 $\mathcal{L}(X_{40})$ = K667

This cubic fulfills the last condition. It contains X_{10}, X_{40}, X_{188}, X_{3057}.

It is a $ps\mathcal{K}$ with respect to the pedal triangle $A'B'C'$ of X_{40} with pseudo-pivot on the line X_1, X_3 and pseudo-isopivot the midpoint of X_8, X_{78}. These two points are unlisted in the current edition of [5].

Since X_{40} is the reflection of I about O, the asymptotes of $\mathcal{L}(X_{40})$ are parallel to the internal bisectors of ABC.

$\mathcal{H}(X_{40})$ decomposes into two perpendicular lines secant at X_9 and passing through X_{2590}, X_{2591}. These are the axis of the Mandart ellipse.

It follows that $\mathcal{L}(X_{40})$ contains the six common points of these axes and the internal bisectors of ABC. See figure 4.

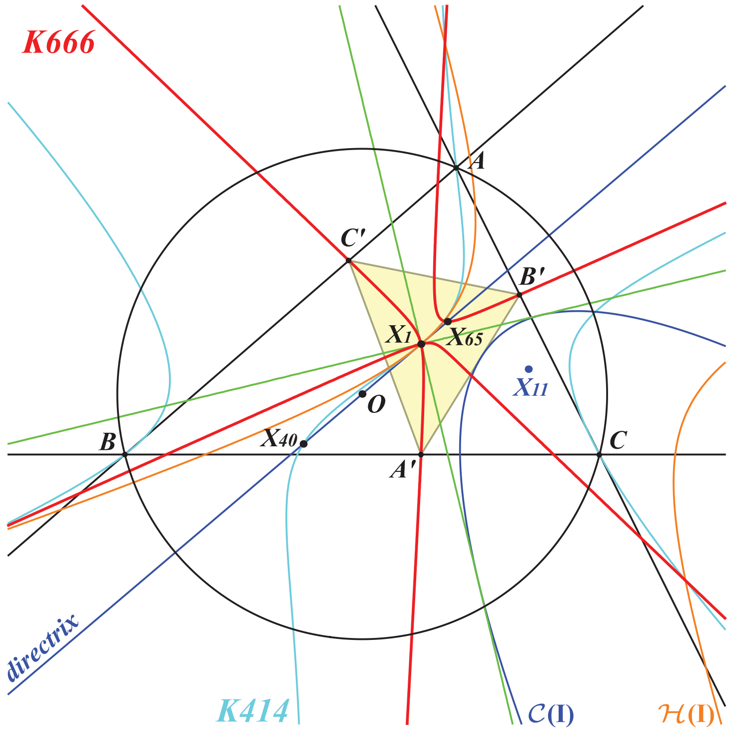

Figure 3: $\mathcal{L}(I) =$ K666

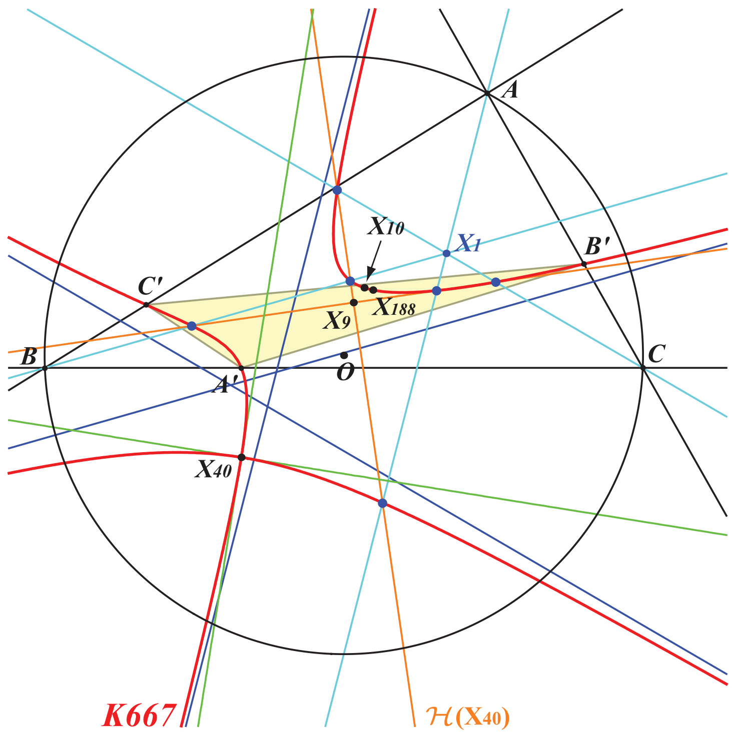

Figure 4: $\mathcal{L}(X_{40}) =$ K667

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