Neuberg Cubics

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Abstract

We characterize the circular pivotal isocubics $\mathcal{K} = p\mathcal{K}(\Omega, P)$ which are Neuberg cubics for some triangle. We take the opportunity to recall the essential properties of circular pivotal cubics and we also examine several pencils of circular cubics related to the Neuberg cubic.

1 Circular Pivotal Cubics

In the plane of the reference triangle ABC, let $\mathcal{K} = p\mathcal{K}(\Omega, P)$ denote the pivotal cubic with pole $\Omega = p : q : r$ and pivot $P = u : v : w$. $\mathcal{K}$ is the locus of point M such that P, M and the $\Omega-$isoconjugate M^* of M are collinear.

The $\Omega-$isoconjugate P^* of P is called the isopivot (or secondary pivot) and $\mathcal{K}$ is also the locus of contacts of tangents drawn through P^* to the circum-conics passing through P. P^* is the tangential of P. For any point M on the cubic, the points P^*, M and P/M (cevian quotient or Ceva conjugate) are collinear.

These tangents pass through the fixed point P/P^* of $\mathcal{K}$ which is the pole of P^* in the pencil of the circum-conics passing through P and therefore the coresidual of A, B, C and P. P/P^* is the tangential of P^*.

$\mathcal{K}$ is said to be a circular cubic when it meets the line at infinity $\mathcal{L}^\infty$ at the same points J_1, J_2 as any circle. J_1, J_2 are called the circular points at infinity or the cyclic points.

When the pivot P is given, $\mathcal{K}$ is circular if and only if one of the three following conditions holds :

1. if P is on $\mathcal{L}^\infty$, the pole must be the Lemoine point K since J_1, J_2 are isogonal conjugates with respect to ABC. Hence $\mathcal{K}$ is an isogonal circular pivotal cubic with respect to ABC and its isopivot P^* lies on the circumcircle. It is the intersection of the cubic with its real asymptote.

   All these cubics form a pencil $\mathcal{P}$ of cubics passing through the nine points A, B, C, I (incenter), I_a, I_b, I_c (excenters), J_1, J_2.

   The most famous is the Neuberg cubic **K001** when P is X_{30}, the infinite point of the Euler line, and P^* is X_{74}.

   Other examples are **K021**, **K269**, **K270** in [8].

2. if $P = H$ (orthocenter of ABC), the pole must be a point on the orthic axis and the isopivot P^* must be on $\mathcal{L}^\infty$.

   Indeed, when M traverses $\mathcal{L}^\infty$, the locus of P/M is the bicevian conic $\mathcal{C}(G, P)$ [1]. Each circular point J_1, J_2 must be the $P-$Ceva conjugate of the other one hence the $P-$Ceva conjugate of $\mathcal{L}^\infty$ must be a circle

[1]this is the conic passing through the vertices of the cevian triangles of G and P.

which turns out to be here the nine point circle i.e. the bicevian conic $\mathcal{C}(G,H)$. Hence, the pivot P must be H.

All these cubics form a pencil $\mathcal{P}'$ of cubics passing through the nine points A, B, C, H, H_a, H_b, H_c (vertices of the orthic triangle), J_1, J_2. They are the isogonal circular pivotal cubics with respect to the orthic triangle. They are invariant under orthoassociation i.e. inversion with respect to the polar circle (see [9]) and also under three other inversions with poles A, B, C swapping H and H_a, H_b, H_c respectively.

For example, when P^* is X_{1154} (the infinite point of the Euler line of the orthic triangle), we obtain **K050**, the Neuberg orthic cubic.

Other examples are **K059**, **K209**, **K334**, **K337**, **K339** in [8]. See also **CL019**.

3. if P is a finite point distinct of H, there is a unique circular pivotal cubic $\mathcal{K}$ with pivot P and, in this case, its isopivot P^* must be the inverse (in the circumcircle) of the isogonal conjugate of P. This property is obviously true for the two first cases above.

These latter cubics are those of main interest in this paper. Indeed, each pencil $\mathcal{P}$ and $\mathcal{P}'$ already contains what we call a Neuberg cubic i.e. an isogonal circular pivotal cubic whose real infinite point is that of the Euler line of a certain triangle inscribed in the cubic. This triangle must be the diagonal triangle of the quadrangle formed by the four fixed points of the isoconjugation.

In the first case, the triangle is ABC itself and the fixed points are the in/excenters of ABC. In the second case, the triangle is the orthic triangle and the fixed points are A, B, C, H.

2 Properties of Circular Pivotal Cubics

In this section, we consider the circular pivotal cubic $\mathcal{K}$ with pivot P, finite point distinct of H. We recall (and complete) without proofs some classical properties of these cubics. See the bibliography.

2.1 First Construction of $\mathcal{K}$

$\mathcal{K}$ contains the vertices of the cevian triangle $P_aP_bP_c$ of P and several remarkable points which we need in the sequel.

Recall that the isopivot P^* is the inverse of the isogonal conjugate of P. It is the tangential of P in the cubic. It follows that the pole Ω is the barycentric product of P and P^*. [2]

[2] The notion of barycentric product of two distinct points P and Q can be defined as follows. Let γ_P be the conic passing through P and the vertices of the anticevian triangle of P which is tangent at P to the line PQ. γ_P is a diagonal conic i.e. the triangle ABC is self-polar in this conic. Define γ_Q in the same way. The barycentric product $P \times Q$ is the intersection of the polar lines of G in the two conics.

Note that the intersection of the polar lines of a point M in these same conics is the isoconjugate of M in the isoconjugation that swaps P and Q.

Moreover, these two conics γ_P and γ_Q meet at the fixed points of this isoconjugation. These points are the (not always real) square roots of the pole $\Omega = P \times Q$.

If P and Q are not distinct, Ω becomes the barycentric square P^2 of P which is the pole of G in the pencil of conics passing through P and the vertices of the anticevian triangle of P.

The classical construction valid for any pivotal cubic can be used here since we know the pivot P and the isopivot P^* : if M is a variable point on the line PP^*, construct $N = M/P$ (cevian quotient of M and P, the perspector of the cevian triangle of M and the anticevian triangle of P). The line PN meets the circum-conic through M and P^* at two isoconjugate points U, U^* on the cubic.

Furthermore, the tangents at U, U^* to the cubic pass through N^*, the second intersection of the line MN with the circum-conic above. Note that :

– N lies on the polar conic of P in the cubic (diagonal conic passing through P, the vertices of the anticevian triangle of P, which is tangent at P to the line PP^*)

– M^* lies on the polar conic of P^* in the cubic (circum-conic passing through P and P^*). M^* is the second intersection with the line PN.

2.2 Other points on $\mathcal{K}$

$\mathcal{K}$ meets $\mathcal{L}^\infty$ at the circular points J_1, J_2 and a third (always real) point J whose $\Omega-$isoconjugate $T = J^*$ is the isogonal conjugate of the complement of P. T is the coresidual of P, P^*, J_1, J_2. In other words, any circle passing through P and P^* meets $\mathcal{K}$ at two other points collinear with T.

In particular, if we consider the degenerate circle $PP^* \cup \mathcal{L}^\infty$, we see that J is the infinite point of the line PT hence the real asymptote $(\mathcal{A})$ of $\mathcal{K}$ is parallel to PT.

The circles passing through A, B, C meet the lines AT, BT, CT again at A_2, B_2, C_2 which also lie on the circles BCP^*, CAP^*, ABP^* respectively.

With $\Omega = p : q : r$, the coordinates of these points are :

$$\begin{aligned} A_2 &= 2p(S_A\, p + S_B\, q + S_C\, r) : U_C : U_B, \\ B_2 &= U_C : 2q(S_A\, p + S_B\, q + S_C\, r) : U_A, \\ C_2 &= U_B : U_A : 2r(S_A\, p + S_B\, q + S_C\, r), \end{aligned}$$

where $U_A = 2a^2qr - c^2q(p - q + r) - b^2r(p + q - r)$, U_B and U_C being defined similarly. [3]

We remark that T is the isotomic conjugate of the point with coordinates $U_A : U_B : U_C$.

When we express the coordinates of A_2, B_2, C_2 with respect to those of P we find :

$$\begin{aligned} A_2 = &- a^2(u+v)(u+w) + b^2u(u+v) + c^2u(u+w) : \\ & b^2(u+v)(u+v+w) : \\ & c^2(u+w)(u+v+w), \end{aligned}$$

$$\begin{aligned} B_2 = \; & a^2(u+v)(u+v+w) : \\ & a^2v(u+v) - b^2(u+v)(v+w) + c^2v(v+w) : \\ & c^2(v+w)(u+v+w), \end{aligned}$$

$$\begin{aligned} C_2 = \; & a^2(u+w)(u+v+w) : \\ & b^2(v+w)(u+v+w) : \\ & a^2w(u+w) + b^2w(v+w) - c^2(u+w)(v+w). \end{aligned}$$

[3] We use the Conway's notations : $2S_O = a^2 + b^2 + c^2$, $2S_A = b^2 + c^2 - a^2$, etc.

The Ω−isoconjugate A_2^* of A_2 is $BC_2 \cap CB_2$, B_2^* and C_2^* are defined similarly. Note that the lines AA_2^*, BB_2^*, CC_2^* are parallel to $(\mathcal{A})$.

The triangles $A_2B_2C_2$ and $P_aP_bP_c$ are perspective at a point Q lying on $\mathcal{K}$ and on the parallel at P^* to $(\mathcal{A})$. The Ω−isoconjugate $S = Q^*$ of Q is the last common point of $\mathcal{K}$ and the circumcircle of ABC. S lies on the lines PQ, RT and on the circle passing through P, P^* and R where $R = P/P^*$.

The lines P_bC_2, P_cB_2, P_aJ also concur on $\mathcal{K}$, two other triads of lines similarly.

2.3 Singular focus of $\mathcal{K}$

The conic passing through the midpoints of $A_2A_2^*$, $B_2B_2^*$, $C_2C_2^*$, PT, P^*Q is the polar conic of the infinite point J. It is a rectangular hyperbola $(\mathcal{H})$ whose center Z is a point on the circumcircle $(\mathcal{C}_2)$ of $A_2B_2C_2$ and on $(\mathcal{A})$. The triangle $A_2B_2C_2$ is self-polar in $(\mathcal{H})$ i.e. $(\mathcal{H})$ is a diagonal conic with respect to $A_2B_2C_2$.

The second intersection of $(\mathcal{C}_2)$ and $(\mathcal{A})$ is the point X where $\mathcal{K}$ meets $(\mathcal{A})$. X is the common tangential of the points A_2, B_2, C_2 and J. X also lies on the line SU where U is the perspector of $P_aP_bP_c$ and $A_2^*B_2^*C_2^*$. Note that $U = P/T$ hence U, T and P^* are collinear. The lines QT and PX meet at X^* on the cubic.

The antipode of X on $(\mathcal{C}_2)$ is the singular focus F of $\mathcal{K}$ hence the polar conic of F is a circle. In general, F is not a point of $\mathcal{K}$.

2.4 Orthic line of $\mathcal{K}$

The locus of points whose polar conic in $\mathcal{K}$ is a rectangular hyperbola is in general [4] a line $(\mathcal{L})$ passing through J hence parallel to the asymptote $(\mathcal{A})$. $(\mathcal{L})$ is called the orthic line [5] of $\mathcal{K}$ and pass through the circumcenter O_2 of $(\mathcal{C}_2)$. It is also the perpendicular bisector of FZ or the homothetic of $(\mathcal{A})$ in the homothety with center F, ratio 1/2.

$(\mathcal{L})$ must meet $\mathcal{K}$ at two other finite points L_1, L_2 which are not necessarily real. The midpoint Y of L_1L_2 obviously lies on $(\mathcal{H})$. These two points L_1, L_2 will have a great importance in our study.

2.5 $\mathcal{K}$ is an isogonal $p\mathcal{K}$. Second construction

$(\mathcal{H})$ meets $\mathcal{K}$ again at four points which are the centers of anallagmaty of $\mathcal{K}$. These points are the incenter E_o and the excenters E_a, E_b, E_c of the triangle $A_2B_2C_2$. This means that the cubic $\mathcal{K}$ is invariant under four inversions, one of them being that of pole E_o swapping A_2 and E_a, B_2 and E_b, C_2 and E_c. The three other inversions with poles E_a, E_b, E_c are defined in the same way.

The coordinates of these points E_o, E_a, E_b, E_c are complicated. When the pole of the cubic is $\Omega = p : q : r$, let

$$T_A = \sqrt{16\Delta^2\, qr + (c^2q - b^2r)^2} = \sqrt{-4S_A^2\, qr + (c^2q + b^2r)^2},$$

[4] If one can find three distinct non collinear points whose polar conic is a rectangular hyperbola then the cubic must be a $\mathcal{K}_{60}^+$ i.e. a cubic with three real concurring asymptotes making 60° angles with one another. This cannot occur in our study.

[5] $(\mathcal{L})$ is the orthic axis of the triangle formed by the asymptotes of a cubic with three real asymptotes.

with $\Delta = \text{area}(ABC)$.

The quantities T_B, T_C being defined similarly and U_A, U_B, U_C being as above, the coordinates of E_o are :

$$\begin{aligned} &2p(S_A\,p + S_B\,q + S_C\,r)T_A + U_C T_B + U_B T_C : \\ &U_C T_A + 2q(S_A\,p + S_B\,q + S_C\,r)T_B + U_A T_C : \\ &U_B T_A + U_A T_B + 2r(S_A\,p + S_B\,q + S_C\,r)T_C. \end{aligned}$$

The other points E_a, E_b, E_c are obtained when T_A, T_B, T_C are respectively replaced by their opposite in the coordinates of E_o. To be more precise, the coordinates of E_a are those of E_o with T_A replaced by $-T_A$.

Remarks :

- the point with barycentric coordinates $T_A : T_B : T_C$ is the barycentric product of the incenter of ABC and the point whose coordinates are the distances from the pivot P to the vertices of ABC.
- $S_A\,p + S_B\,q + S_C\,r = 0$ if and only if Ω lies on the orthic axis in which case the pivot P must be H. One of the points E_o, E_a, E_b, E_c is H and the other are the vertices of ABC. When ABC is acute angled, $E_o = H$, $E_a = A$, etc.

It follows that $\mathcal{K}$ is an isogonal $p\mathcal{K}$ with respect to the triangle $A_2B_2C_2$ with pivot J at infinity and isopivot X on $(\mathcal{C}_2)$.

This gives another construction valid for any isogonal circular cubic. Let γ be a circle passing through E_o and centered at ω on the perpendicular at E_o to $(\mathcal{L})$. The perpendicular at X to the line $F\omega$ meets γ at two points on the cubic.

2.6 Examples

2.6.1 The Droussent cubic K008

The first example to illustrate these properties is the Droussent cubic, the only isotomic circular $p\mathcal{K}$. See [6], [8] and figure 1.

The pivot P is X_{316} and the isopivot P^* is X_{67}. The other points on the cubic are $J = X_{524}$, $T = X_{671}$, $Q = X_{858}$, $Q^* = X_{2373}$.

The real asymptote $(\mathcal{A})$ is the parallel to the line GK at the Parry point X_{111}.

F, X, O_2, L_1, L_2 are not mentioned in the current edition of [12].

– The Droussent focus F is the intersection of the lines X_3X_{126}, X_5X_{111}, $X_{30}X_{1296}$, etc.

– X is the intersection of the lines $X_{111}X_{524}$ (the real asymptote) and X_4E_{620} where E_{620} is the anticomplement of X_{111}. E_{620} and its isotomic conjugate E_{635} are two points of **K008**.

The orthic line $\mathcal{L}$ is the parallel to the line GK at the nine point center X_5. The two points L_1, L_2 are always real but their coordinates are rather complicated. They lie on the circum-conic passing through X_{2373} which is the isogonal conjugate (with respect to ABC) of the line $X_{575}X_{2393}$.

In this case, the coordinates of A_2 are :

$$-2S_O : 2S_C - c^2 : 2S_B - b^2,$$

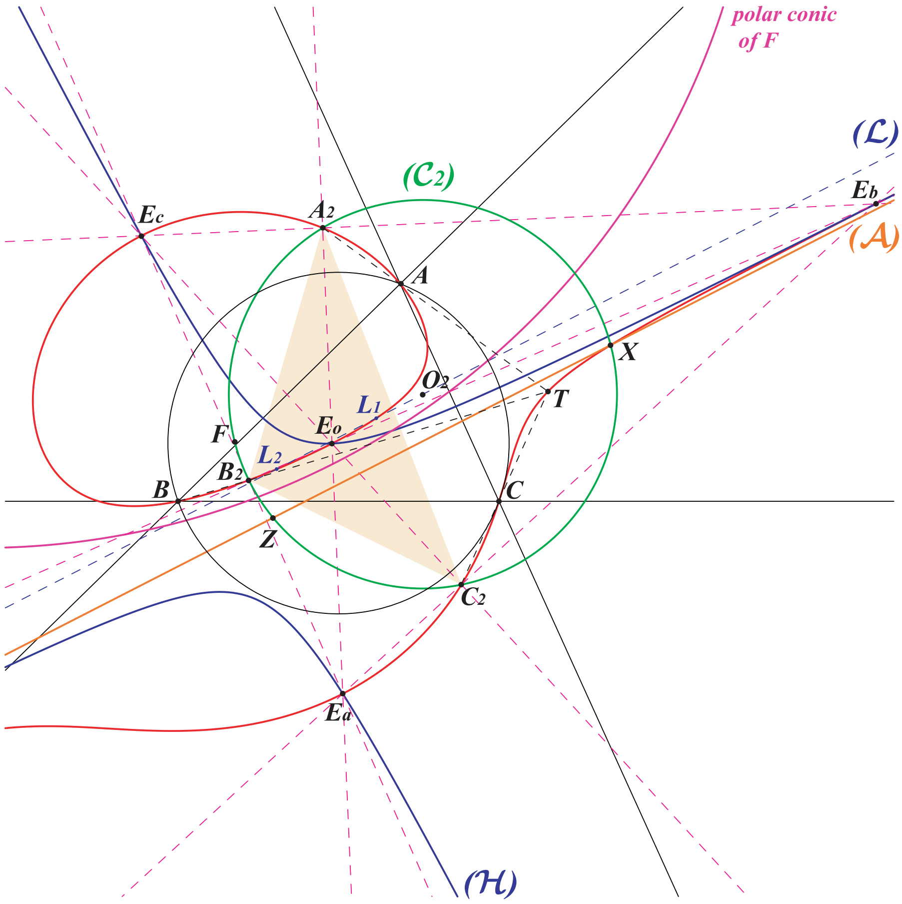

Figure 1: The Droussent cubic **K008**

B_2, C_2 similarly.

The excenters E_a, E_b, E_c of triangle $A_2B_2C_2$ are the extraversions of its incenter E_o with coordinates :

$$\begin{aligned}
-2S_OT_A + (2S_C - c^2)T_B + (2S_B - b^2)T_C :\\
(2S_C - c^2)T_A - 2S_OT_B + (2S_A - a^2)T_C :\\
(2S_B - b^2)T_A + (2S_A - a^2)T_B - 2S_OT_C,
\end{aligned}$$

where $T_A = a\sqrt{2b^2 + 2c^2 - a^2}$, T_B and T_C similarly.

We remark that the triangles ABC and $E_aE_bE_c$ are perspective at the isotomic conjugate tE_o of E_o and the triangles $A_2B_2C_2$ and $E_aE_bE_c$ are perspective at E_o. In fact, any two of the six triangles ABC, $G_aG_bG_c$, $P_aP_bP_c$, $A_2B_2C_2$, $E_aE_bE_c$ and $A_2^*B_2^*C_2^*$ are perspective.

The polar conic $(\mathcal{H})$ of J contains X_3, X_{524}, X_{599}, X_{1499} and the four in/excenters E_o, E_a, E_b, E_c. Recall that these points are the centers of anallagmaty of the cubic.

2.6.2 The Ki cubic K073

The second example is the inverse (in the circumcircle) of the Neuberg cubic namely the Ki cubic **K073** $= p\mathcal{K}(X_{50}, X_3)$ which is also the isogonal transform of the Kn cubic **K060** $= p\mathcal{K}(X_{1989}, X_{265})$.

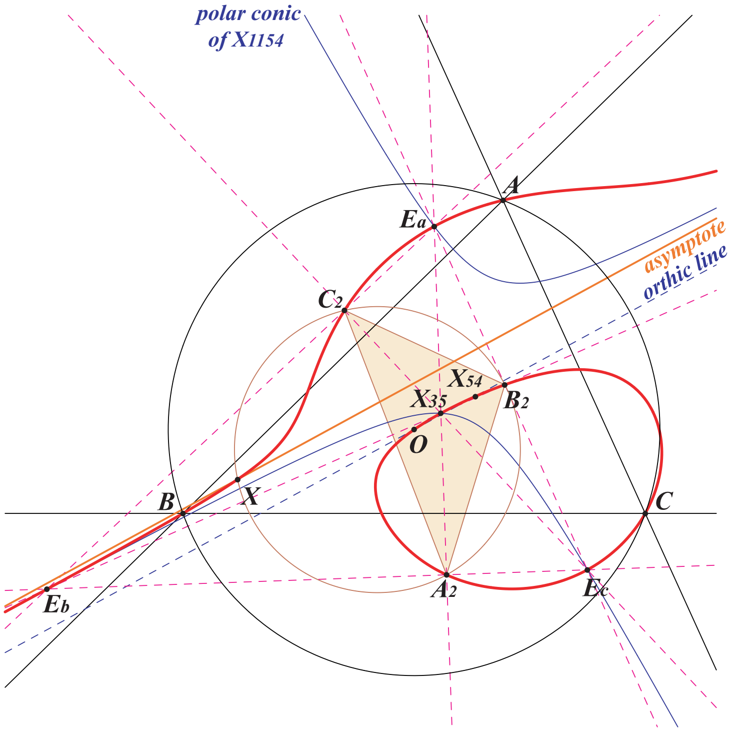

Figure 2: The cubic **K073**

The points A_2, B_2, C_2 are the inverses of the reflections of A, B, C in the sidelines of ABC. The in/excenters E_o, E_a, E_b, E_c of $A_2B_2C_2$ are X_{35} and its extraversions. See figure 2.

Most of the points are here clearly identifiable : $P = X_3$, $P^* = X_{186}$, $J = X_{1154}$, $S = X_{74}$, $Q = X_{1511}$, $T = X_{54}$. L_1 and L_2 are X_3 and X_{54} on the orthic line.

2.7 Deferent parabolas. Third construction

Any circular cubic can be seen as the envelope of circles centered on a fixed parabola called the *deferent parabola* ("déférente" in French) and orthogonal to a fixed circle called the *director circle*.

The focus of the parabola is the singular focus F of the cubic and its directrix must be parallel to the real asymptote of the cubic. The director circle must have its center at one of the centers of anallagmaty of the cubic. Hence there are four parabolas $\mathcal{P}_x$ and four circles $\mathcal{C}_x$, $x \in \{o, a, b, c\}$, with centers the points E_o, E_a, E_b, E_c. Note that the circles are not all real.

If we choose $\mathcal{C}_o$ with center E_o as director circle, the construction of the directrix $\mathcal{D}_o$ of $\mathcal{P}_o$ is easy to realize : reflect F in the perpendicular bisector of A_2E_a (or B_2E_b or C_2E_c) and draw the parallel through this point to the real asymptote to obtain $\mathcal{D}_o$. The construction of $\mathcal{P}_o$ follows immediately.

Now, let M be a variable point on $\mathcal{P}_o$ and let $\mathcal{T}_M$ be the tangent at M to $\mathcal{P}_o$ (this is the perpendicular bisector of F and the projection of M on the directrix). The perpendicular at E_o to $\mathcal{T}_M$ meets the circle with center M and orthogonal to $\mathcal{C}_o$ at two points on the cubic and this latter circle is bitangent at these two points to the cubic.

The three other parabolas give three other families of bitangent circles to the cubic.

We illustrate these properties with the Neuberg cubic itself since the configuration is quite simple. The points E_o, E_a, E_b, E_c are the in/excenters of ABC and $A_2B_2C_2$ is ABC as seen above. F is X_{110}, the focus of the Kiepert parabola and the real asymptote of the cubic is the Euler line.

The directrices of the four parabolas (with same focus X_{110}) are the reflections about the Euler line of the four parallels at the in/excenters to this same Euler line. For example, $\mathcal{D}_o$ is the line through X_{30}, X_{40}, X_{191}, etc. See figure 3.

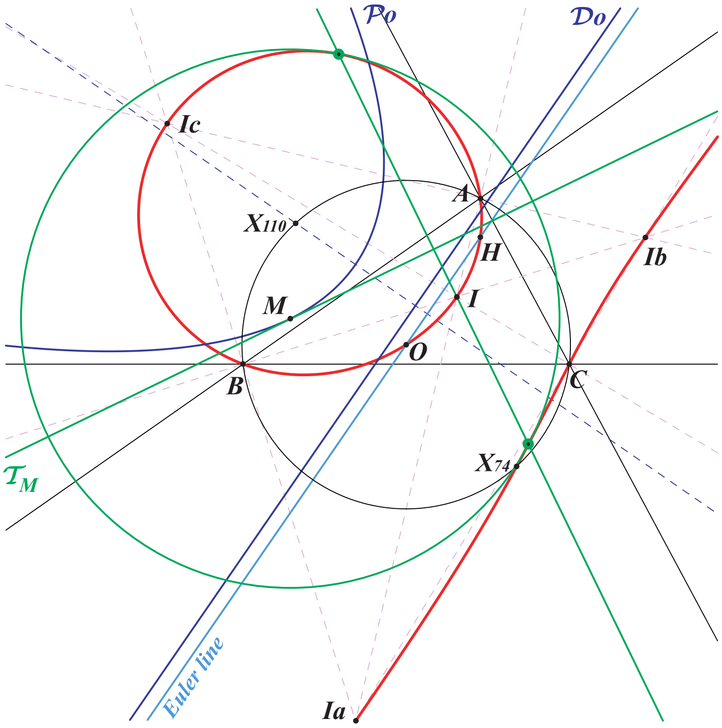

Figure 3: The Neuberg cubic and a deferent parabola

Naturally, these four directrices meet the Neuberg cubic at eight (not all real) two by two isogonal conjugate points of the cubic.

3 Neuberg Cubics

The orthic line of the Neuberg cubic is the Euler line of ABC and the points L_1, L_2 are the circumcenter O and the orthocenter H of ABC.

We already know that the orthic line $(\mathcal{L})$ of $\mathcal{K}$ passes through O_2, the circumcenter of $(\mathcal{C}_2)$, which is in general not a point of $\mathcal{K}$. When O_2 lies on $\mathcal{K}$, its isogonal conjugate H_2 (with respect to $A_2B_2C_2$) is obviously the orthocenter of $A_2B_2C_2$ and must also lie on $\mathcal{K}$. The cubic is a Neuberg cubic with respect to $A_2B_2C_2$ if and only if L_1, L_2 are the orthocenter and circumcenter of $A_2B_2C_2$. A (tedious) computation shows that the pivot P of $\mathcal{K}$ must lie on a quadricircular circum-nonic **Q072** in ABC. See figure 4.

Q072 also contains :

– G_a, G_b, G_c (vertices of the antimedial triangle) but, in this case, the isopivot is a midpoint of ABC and the cubic degenerates,

– H, giving the Neuberg orthic cubic **K050** which is also the orthopivotal cubic $\mathcal{O}(X_4)$. See [9].

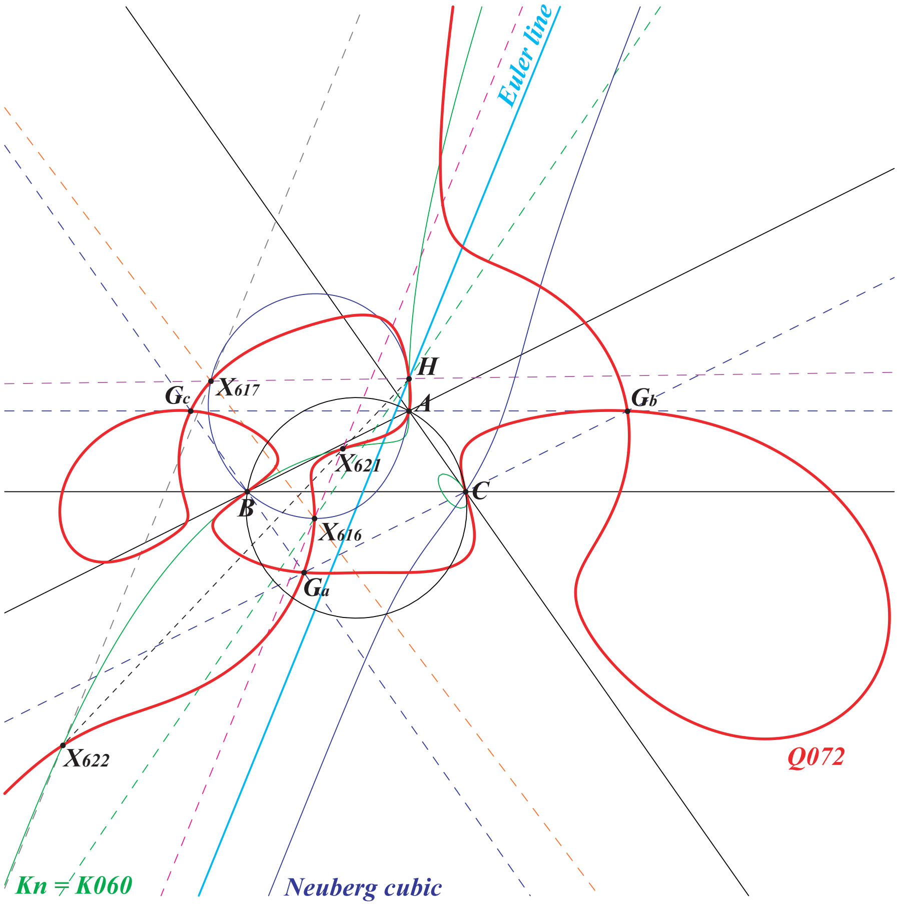

Figure 4: The quadricircular circum-nonic **Q072**

– X_{30} (infinite point of the Euler line of ABC), giving the Neuberg cubic **K001** itself, which is also the orthopivotal cubic $\mathcal{O}(X_3)$.

– X_{616}, X_{617} (anticomplements of the Fermat points X_{13}, X_{14}), giving the cubics **K438a** and **K438b**,

– X_{621}, X_{622} (anticomplements of the isodynamic points X_{15}, X_{16}), giving two other orthopivotal cubics **K066b** $= p\mathcal{K}(X_{395}, X_{621}) = \mathcal{O}(X_{627})$ and **K066a** $= p\mathcal{K}(X_{396}, X_{622}) = \mathcal{O}(X_{628})$ respectively.

The complement of **Q072** is another quadricircular circum-nonic which contains X_3, X_{13}, X_{14}, X_{15}, X_{16}, X_{30} and the midpoints of ABC. Hence, for any point M on this latter nonic, the pivotal cubic $\mathcal{K}$ with pivot $P = aM$ (anticomplement of M), isopivot $P^* = igaM$ (inverse of the isogonal conjugate of P) is a Neuberg cubic for the triangle $A_2B_2C_2$.

It follows that $\mathcal{K}$ contains the counterparts of all the points of the Neuberg cubic in $A_2B_2C_2$ (more than one hundred are known, see [8], table 19), and, in particular, the apices of the equilateral triangles A_e, B_e, C_e or A_i, B_i, C_i drawn externally or internally on the sides of $A_2B_2C_2$. Naturally, the corresponding perspectors will give the Fermat points F_e, F_i of $A_2B_2C_2$ and their isogonal conjugates (with respect to $A_2B_2C_2$) will give the isodynamic points I_e, I_i of $A_2B_2C_2$.

3.1 The Neuberg orthic cubic K050

The triangle $A_2B_2C_2$ is inscribed in ABC (A_2 on BC, etc) if and only if the pivot P is H. In this case, $A_2B_2C_2$ is the orthic triangle and the corresponding cubic is **K050**, the Neuberg orthic cubic, passing through the centers X_4, X_5, X_{15}, X_{16}, X_{52}, X_{128}, X_{186}, X_{1154}, X_{1263}, X_{2383}, X_{2902}, X_{2903}, X_{2914}. See figure 5.

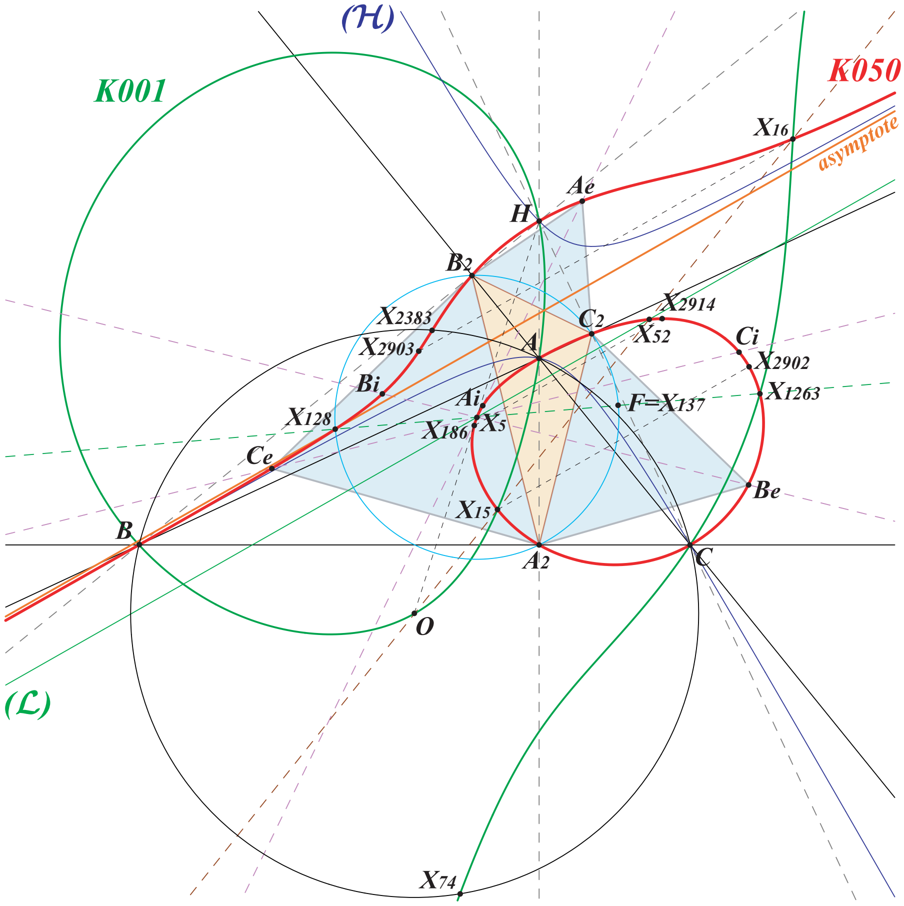

Figure 5: The Neuberg orthic cubic **K050**

The nine common points of **K001** and **K050** are known : A, B, C, the circular points J_1, J_2, the orthocenter H, the isodynamic points X_{15}, X_{16} and X_{1263} (the Parry reflection point of the orthic triangle). These nine points form the basis of a pencil of circular circum-cubics we call the Neuberg pencil. See §5 for further details.

The singular focus is X_{137} that is the focus of the Kiepert parabola of the orthic triangle. The corresponding directrix is naturally X_5X_{51}, the Euler line of the orthic triangle.

K050 contains the counterparts of the centers on **K001** with respect to the orthic triangle. The following table gives a small selection of these points M on **K001** and their counterparts M' on **K050**. Several of these centers are not mentioned in the current edition of [12], in particular the Fermat and isodynamic points X'_{13}, X'_{14}, X'_{15}, X'_{15} of the orthic triangle.

Remarks :

– X_5X_{52} is the Euler line of the orthic triangle and X_{1154} is its infinite point.

Table 1: Corresponding centers on **K001** and **K050**

M on **K001**	M' on **K050**	notes
in/excenters	X_4, A, B, C	
X_3	X_5	circumcenters
X_4	X_{52}	orthocenters
X_{13}	X'_{13}	$X'_{13} = X_{16}X_{186} \cap X_{53}X_{1263}$
X_{14}	X'_{14}	$X'_{14} = X_{15}X_{186} \cap X_{53}X_{1263}$
X_{15}	X'_{15}	$X'_{15} = X_4X_{15} \cap X_5X_{53}$
X_{16}	X'_{16}	$X'_{16} = X_4X_{16} \cap X_5X_{53}$
X_{30}	X_{1154}	infinite points
X_{74}	X_{128}	intersections with the asymptote
X_{399}	X_{1263}	Parry reflection points
X_{1276}	X_{15}	see (1)
X_{1277}	X_{16}	see (1)

– X_5X_{53} is the Brocard axis of the orthic triangle.
– $X_{53}X_{1263}$ is the Fermat axis of the orthic triangle.
– (1) this is true when the incenter of the orthic triangle is H i.e. when ABC is acute angle.

3.2 The cubics K066a and K066b

These are the two cubics obtained when the pivot P is X_{621} for **K066b** $= p\mathcal{K}(X_{395}, X_{621})$ and X_{622} for **K066a** $= p\mathcal{K}(X_{396}, X_{622})$. X_{621} and X_{622} are the anticomplements of the isodynamic points X_{15} and X_{16} respectively. Both cubics are orthopivotal cubics as seen in [9]. See figure 6.

The vertices of $A_2B_2C_2$ are the intersections of the perpendiculars at X_5 to the sidelines of ABC with the cevian lines of the Fermat points, X_{14} for **K066a** and X_{13} for **K066b**.

For both cubics **K066a** and **K066b**, the triangle $A_2B_2C_2$ has the same orientation as ABC itself. Indeed, their algebraic areas are $\Delta(5+\sqrt{3}\cot\omega)/8$ and $\Delta(5-\sqrt{3}\cot\omega)/8$ respectively, where Δ is the area of ABC and ω its Brocard angle. These are positive multiples of Δ.

Their corresponding circumcenters O_2 of $A_2B_2C_2$ are X_{17} for **K066a** and X_{18} for **K066b** and the corresponding orthocenters H_2 of $A_2B_2C_2$ are X_{628} for **K066a** and X_{627} for **K066b**.

Another remarkable thing to note is that the two triangles $A_2B_2C_2$ have the same centroid G, that of ABC, and are orthologic with ABC, one of the center of orthology being the nine point center X_5 and the other being one of the Napoleon points, X_{17} for **K066a** and X_{18} for **K066b**. It follows that the Euler lines of these triangles are GX_{17} and GX_{18}.

Furthermore, ABC and $A_2B_2C_2$ also share one of their Fermat points namely X_{14} for **K066b** and X_{13} for **K066a**.

The apices of the equilateral triangle $A_iB_iC_i$ drawn externally on the sides of $A_2B_2C_2$ are A, B, C for **K066b** and the apices of the equilateral triangle $A_eB_eC_e$ drawn internally on the sides of $A_2B_2C_2$ are A, B, C for **K066a**.

These two cubics have already seven known common points namely A, B, C, J_1, J_2, X_{13}, X_{14} and must meet at two other (always real) points E_1,

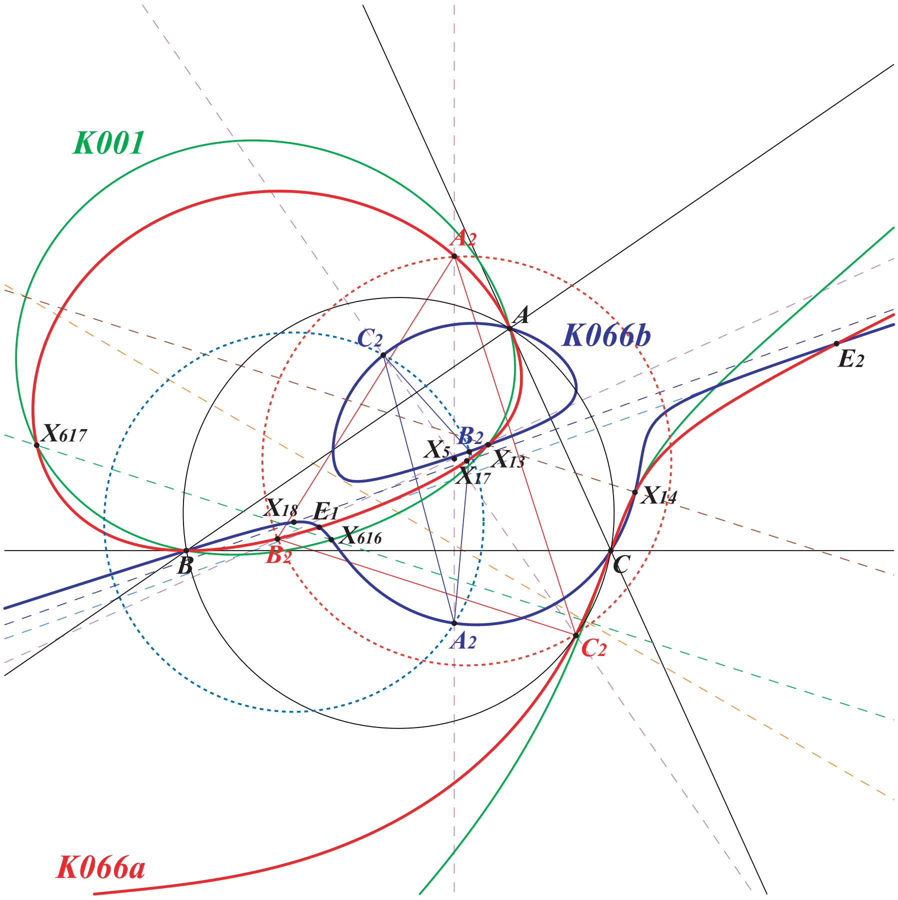

Figure 6: The cubics **K066a** and **K066b**

E_2 which lie on the line passing through X_{54}, X_{69} and on the circle which is its antiorthocorrespondent. This circle is orthogonal to the polar circle and its center is the complement of the isotomic conjugate of the trilinear pole of the line through X_{54}, X_{69}.

K066a and **K066b** generate a pencil of circular circum-cubics which are the orthopivotal cubics with orthopivot on the line through X_{54}, X_{69}. This pencil contains **K112** (the third $p\mathcal{K}$ of the pencil, an inversible cubic), **K292** (the only cubic which contains the isodynamic points X_{15}, X_{16}) and **K442** (the orthopivotal cubic with orthopivot X_{69}). See [8] and [9].

The singular focus F of each cubic lies on the circle passing through X_2, X_{23}, X_{126}, X_{137}.

The following table gives the centers (in the current edition of ETC) lying on these cubics and the singular focus F. The nine already mentioned points are omitted.

Notes :

– X^*_{1337} and X^*_{1338} are the isogonal conjugates of X_{1337} and X_{1338}. These four points lie on the Neuberg cubic.

– $E_{628} = X_{14}, X_{617} \cap X_{17}, X_{622} \cap X_{532}, X_{618}$.

– $E_{629} = X_{13}, X_{616} \cap X_{18}, X_{621} \cap X_{533}, X_{619}$.

Table 2: The pencil generated by **K066a** and **K066b**

cubic	centers	F	remark
K066a	X_{17}, X_{532}, X_{617}, X_{618}, X_{622}, X_{627}, X^*_{1337}, E_{628}		$p\mathcal{K}$
K066b	X_{18}, X_{533}, X_{616}, X_{619}, X_{621}, X_{628}, X^*_{1338}, E_{629}		$p\mathcal{K}$
K112	X_3, X_{54}, X_{96}, X_{265}, X_{539}, X_{1141}, X_{1157}		$p\mathcal{K}$
K292	X_{15}, X_{16}, X_{98}, X_{182}, X_{542}	X_{23}	
K442	X_2, X_{69}, X_{524}, X_{2373}	X_{126}	

3.3 The cubics K438a and K438b

These are the two cubics obtained when P is X_{616} for **K438a** and X_{617} for **K438b**. X_{616} and X_{617} are the anticomplements of the Fermat points X_{13} and X_{14} respectively. See figure 7.

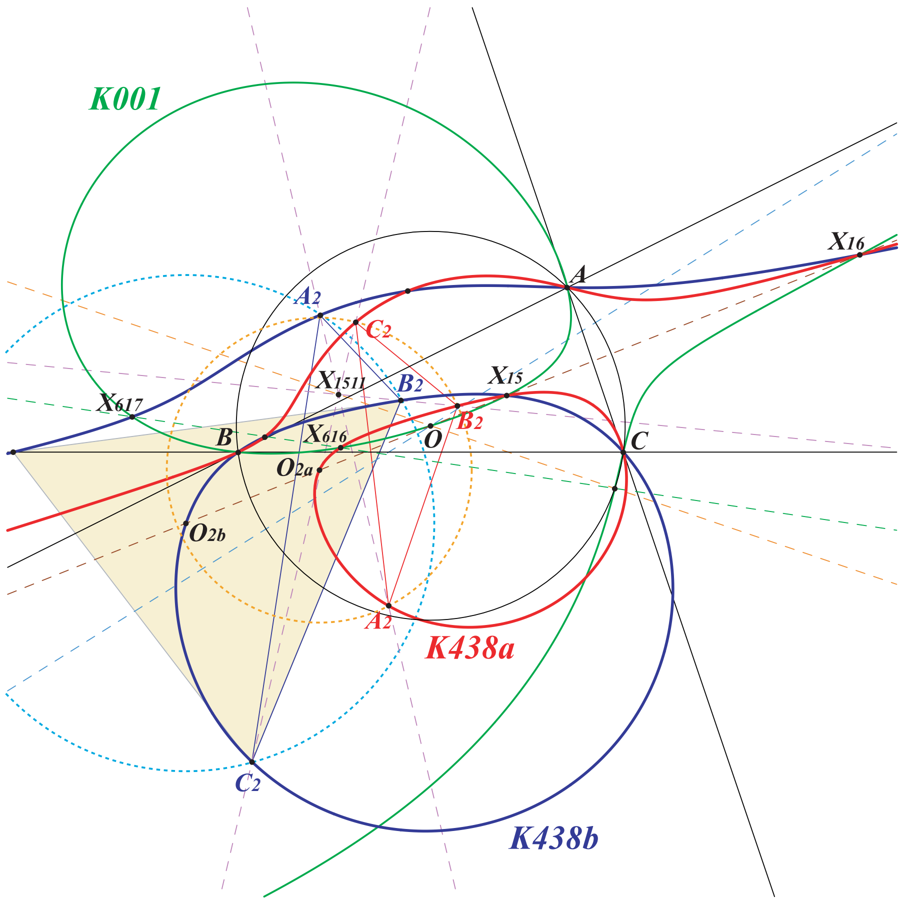

Figure 7: The cubics **K438a** and **K438b**

The triangle $A_2B_2C_2$ has the same orientation as ABC itself for **K438a** and the opposite orientation for **K438b**. Indeed, their algebraic areas are $\Delta(-1+\sqrt{3}\cot\omega)/8$ and $\Delta(-1-\sqrt{3}\cot\omega)/8$. These are positive and negative multiples of Δ.

These two triangles are perspective with ABC at the isodynamic points X_{15} for **K438a** (red triangle on the figure) and X_{16} for **K438b** (blue triangle on the figure). They are also themselves perspective at X_{1511} (midpoint of

X_3, X_{110}).

The vertices B_2 and C_2 for both triangles lie on two lines passing through the midpoint of BC and making $60°$ angles with the sideline BC.

The triangles $A_2B_2C_2$ are also perspective with the cevian triangle of the corresponding pivot and the perspector is a point on the corresponding cubic. Furthermore, the lines joining the corresponding vertices of a triangle $A_2B_2C_2$ and the vertices of a cevian triangle also make $60°$ angles with one another. See figure 8 where several of these $60°$ angles are represented in yellow. The light blue triangle is the cevian triangle $P_aP_bP_c$ of X_{617} .

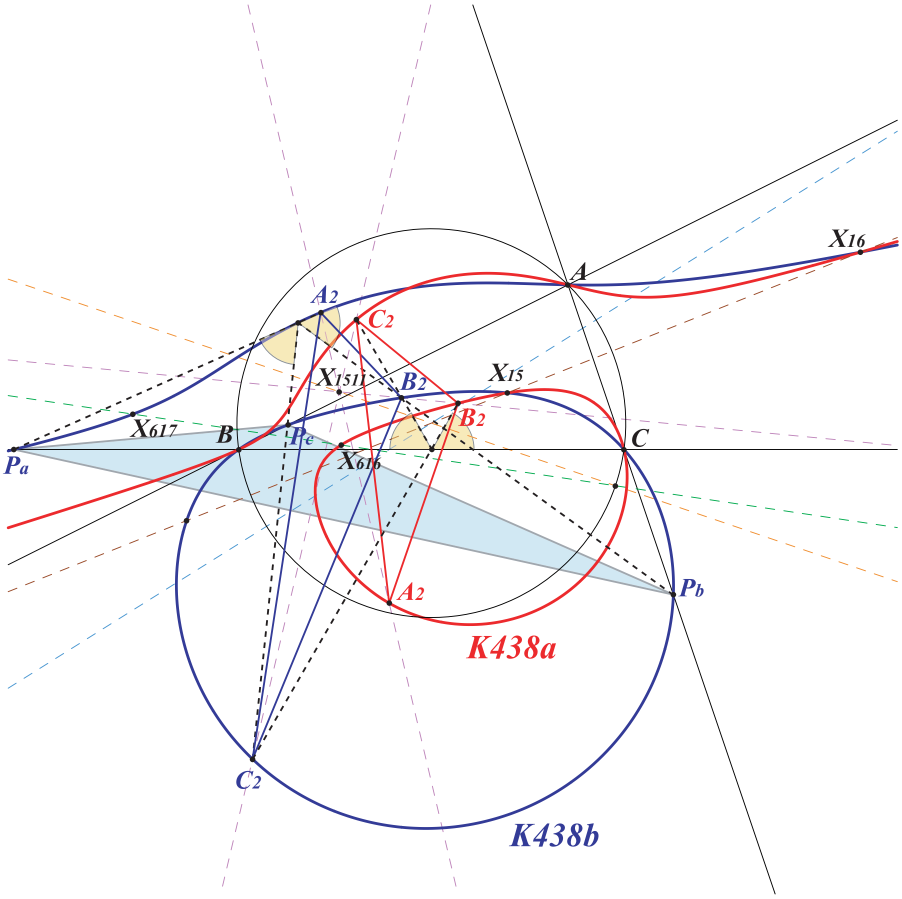

Figure 8: The cubics **K438a** and **K438b** and the triangles $A_2B_2C_2$

The circumcenters O_{2a} and O_{2b} of these triangles lie on the Brocard axis.

In both cases, the apices of the equilateral triangle A_i, B_i, C_i drawn internally on the sides of $A_2B_2C_2$ are the vertices of the cevian triangle of the pivot. The figure 7 shows one of these equilateral triangles, namely the one with vertex the trace of the cevian of X_{617} on the sideline BC.

The ETC centers lying on these cubics are :
K438a : X_{15}, X_{16}, X_{532}, X_{616}, X_{618} (complement of X_{13}).
K438b : X_{15}, X_{16}, X_{533}, X_{617}, X_{619} (complement of X_{14}).

These two cubics generate a pencil containing a third pivotal cubic which is $p\mathcal{K}(X_{1511} \times X_{1138}, X_{1138})$ i.e. the pivotal cubic with pivot X_{1138} (isogonal conjugate of the Parry reflection point X_{399}) and isopivot X_{1511}. This passes through X_{15}, X_{16}, X_{30}, X_{1138}, X_{1511}.

4 Special triangles $A_2B_2C_2$

We already know that :
– ABC and $P_aP_bP_c$ are perspective at P,
– $A_2B_2C_2$ and ABC are perspective at T,
– $A_2B_2C_2$ and $P_aP_bP_c$ are perspective at Q,
these three points P, T, Q lying on the cubic $\mathcal{K}$.

With $P = u : v : w$ not on $\mathcal{L}^\infty$, we have

$$A_2 = -(u+v)(u+w)a^2 + u(u+v)b^2 + u(u+w)c^2 : \\ (u+v)(u+v+w)b^2 : (u+w)(u+v+w)c^2,$$

B_2 and C_2 likewise.

4.1 Cevian triangles $A_2B_2C_2$

$A_2B_2C_2$ is a cevian triangle if and only if its vertices lie on the sidelines of ABC hence if and only if P is the orthocenter H of ABC. In this case, the points P and T coincide and $A_2B_2C_2$ is the orthic triangle of ABC. It follows that the isopivot lies at infinity and we obtain the isogonal circular cubics with respect to the orthic triangle as seen at the beginning.

4.2 Anticevian triangles $A_2B_2C_2$

$A_2B_2C_2$ is an anticevian triangle if and only if the points A, B_2, C_2, etc, are collinear. These three conditions show that P must lie on the circle $\mathcal{C}(H, 2R)$, the anticomplement of the circumcircle. It follows that the pole is a point on the inscribed Steiner ellipse.

In this case, $A_2B_2C_2$ is the anticevian triangle of the real infinite point J of the cubic thus $T = J$. P is now the intersection of the cubic with its real asymptote i.e. $P = X$. Note that A, B_2, C_2 lie on the sidelines of the medial triangle.

When J traverses the line at infinity, the envelope of the real asymptote is a deltoid tritangent to $\mathcal{C}(H, 2R)$ and the locus of the orthocenter H_2 of $A_2B_2C_2$ is a nodal cubic with node X_{20}, passing through the in/excenters of ABC, X_{1158} and its extraversions. The three asymptotes are parallel to the altitudes of ABC and concur at the midpoint X_{550} of O, X_{20}.

The figure 9 presents $p\mathcal{K}(X_{1086}, X_{150})$ – which is an example of such cubic – together with the locus of H_2. The orthocenter H_2 of $A_2B_2C_2$ is here the incenter X_1 of ABC. The pivot P is X_{150}, the anticomplement of X_{101}.

4.3 Orthologic triangles

The triangles ABC and $A_2B_2C_2$ are orthologic if and only if either :

– P is on the line at infinity in which case Ω lies on the orthic axis, but the two triangles ABC and $A_2B_2C_2$ here coincide,

– P is on a bicircular quintic $\mathcal{Q}$ passing through X_4, X_8 (and its extraversions N_a, N_b, N_c), X_{30}, X_{621}, X_{622}, the foci of the Steiner ellipse, the common points of the Lucas cubic and the circle $\mathcal{C}(H, 2R)$. See figure 10. This quintic is self-inverse in this latter circle since it is the anticomplement of the inversible bicircular quintic **Q037**, see [8]. In this case Ω lies on another quintic passing through X_6, X_{37}, X_{44}, X_{395}, X_{396}, X_{3003}.

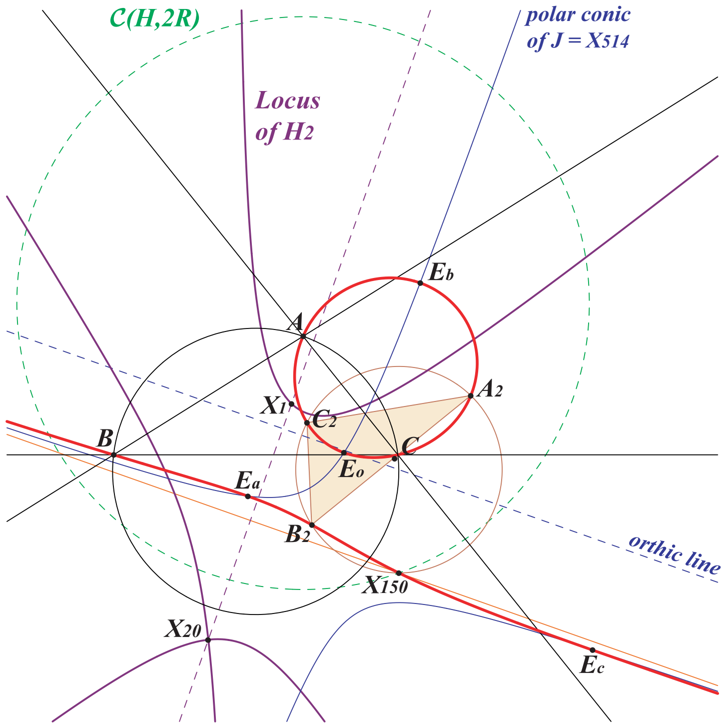

Figure 9: The cubic $p\mathcal{K}(X_{1086}, X_{150})$ and the locus of H_2

This gives the Neuberg cubic **K001** $= p\mathcal{K}(X_6, X_{30})$, **K338** $= p\mathcal{K}(X_{44}, X_8)$, **K066b** $= p\mathcal{K}(X_{395}, X_{621})$, **K066a** $= p\mathcal{K}(X_{396}, X_{622})$, **K339** $=p\mathcal{K}(X_{3003}, X_4)$ and another interesting cubic : $p\mathcal{K}(X_{37}, aX_{36})$, (where aX_{36} is the anticomplement of X_{36}) passing through X_4, X_{12}, X_{21}, X_{72}, X_{80}, X_{191}, X_{502}, X_{758} for which ABC and $A_2B_2C_2$ are orthologic at X_{10} and X_{21}.

5 The Neuberg pencil

We recall that the Neuberg pencil $\mathcal{N}$ is the pencil of circular circumcubics generated by the Neuberg cubic **K001** and the Neuberg orthic cubic **K050**. All these cubics contain A, B, C, J_1, J_2, H, X_{15}, X_{16} and X_{1263}. Their orthic line always contains the nine point center X_5.

Their singular foci lie on the circle passing through X_5, X_{23}, X_{110} (that of **K001**), X_{114}, X_{137} (that of **K050**).

The real asymptote envelopes a deltoid tritangent to the circle passing through X_5, X_{115}, X_{128}, X_{265}, the reflection of the circle above in X_5. See figure 11.

$\mathcal{N}$ contains several interesting cubics which are examined in the following paragraphs.

5.1 The cubic K439

The pencil $\mathcal{N}$ contains a third $p\mathcal{K}$ which is **K439** $= p\mathcal{K}(X_{54}\times X_{1263}, X_{1263})$ i.e. the pivotal cubic with pivot X_{1263} (the Parry reflection point of the orthic triangle) and isopivot X_{54} (the Kosnita point, isogonal conjugate of the nine point center X_5). See figure 12.

K439 contains the ETC centers X_4, X_{15}, X_{16}, X_{54}, X_{140}, X_{1263}, X_{2070} and the isogonal conjugate E_{557} of X_{195}.

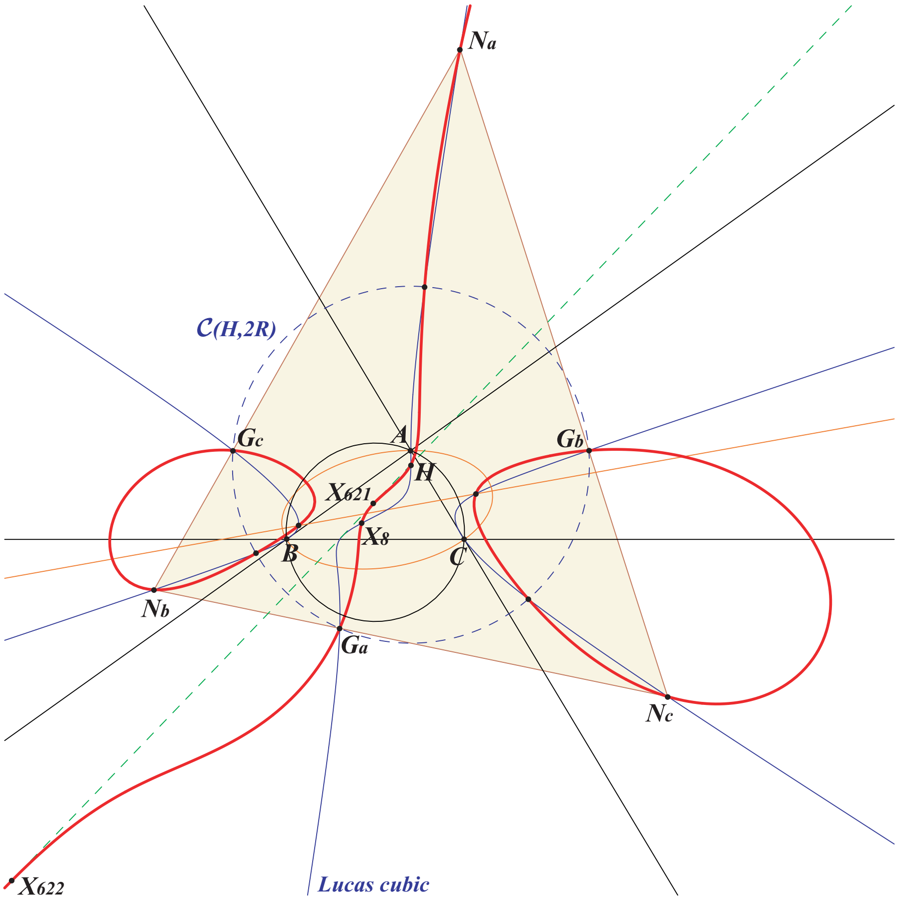

Figure 10: The quintic $\mathcal{Q}$ together with the Lucas cubic

The third point on the Brocard axis is E_{216} on the lines X_5X_{195}, $X_{49}X_{51}$, $X_{54}X_{143}$, etc.

The last point P on the circumcircle is its intersection other than X_{930} with the line $X_{140}X_{1263}$.

The singular focus is the second intersection of the line X_5X_{930} with the circle cited above.

K439 is the isogonal transform of the orthopivotal cubic **K067**.

5.2 The cubic K304

K304 is the cubic of $\mathcal{N}$ passing through the centroid G of ABC. It is the isodynamic Droz-Farny cubic DF(Q) where Q is the intersection of the lines GK and HX_{1263}. See [8], **CL019**.

K304 contains the ETC centers X_2, X_4, X_{15}, X_{16}, X_{524}, X_{576}, X_{671}, X_{1263}. See figure 13.

The orthic line of **K304** is the parallel at X_5 to the line GK, meeting the cubic at G, X_{524} and X_{576}.

The singular focus is the second intersection of the line $X_{23}X_{111}$ with the circle cited above.

5.3 The cubic K440

K440 is a remarkable $\mathcal{K}^+$ i.e. a cubic with three concurring asymptotes at its singular focus X_5. Thus, the singular focus lies on the real asymptote

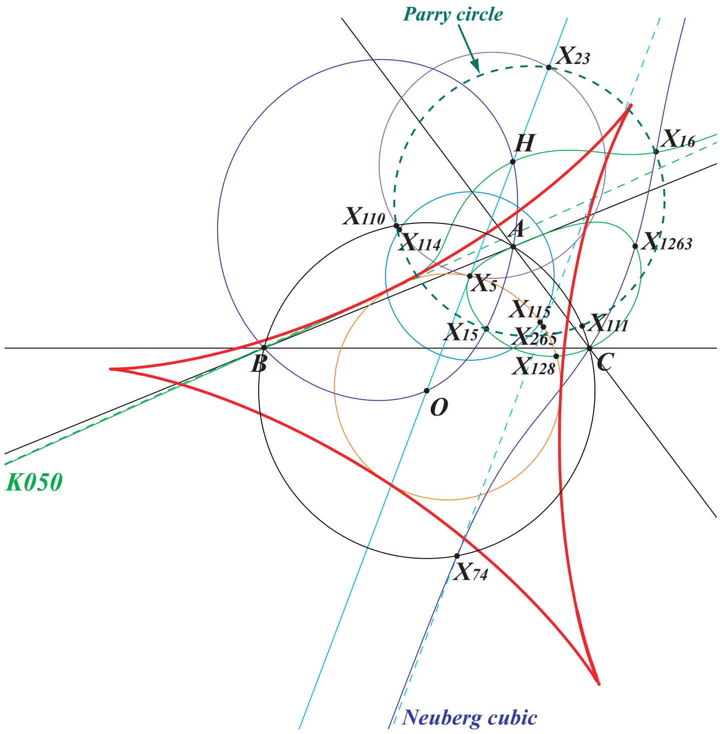

Figure 11: The deltoid of the Neuberg pencil

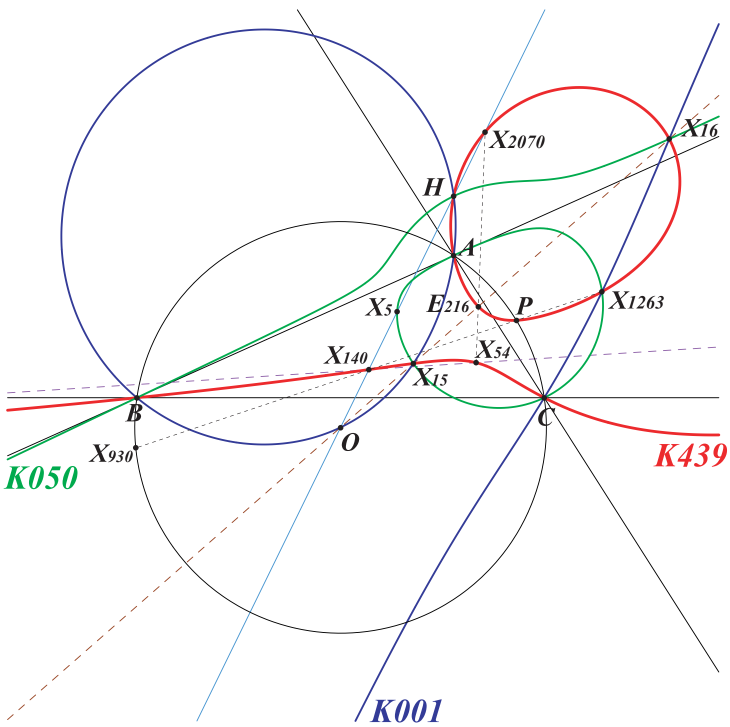

Figure 12: The cubic **K439** together with **K001** and **K050**

which is the parallel at X_5 to the Brocard axis. The orthic line is here the real asymptote itself.

It follows that the polar conic of X_5 must decompose into the line at infinity and a line through X_{570} (a point on the Brocard axis) which is the

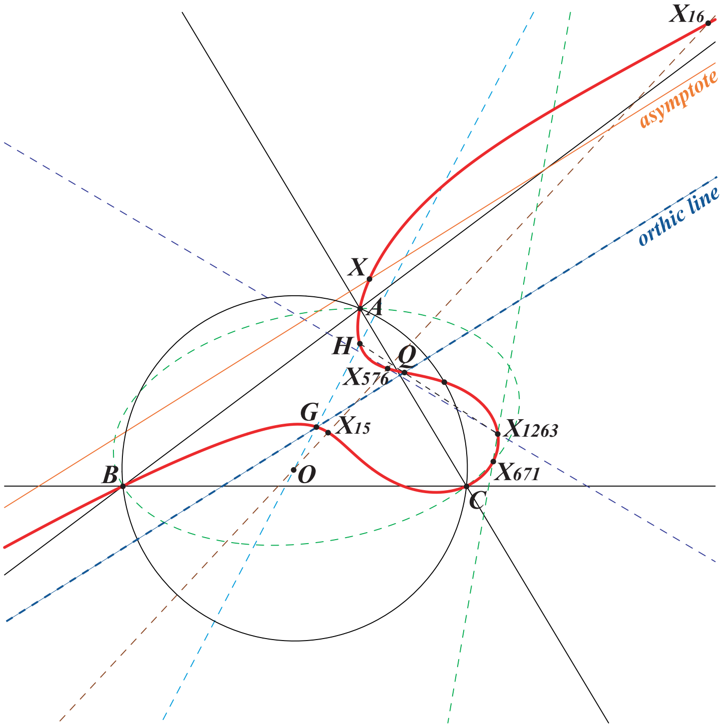

Figure 13: The cubic **K304**

harmonic polar of X_5. See figure 14.

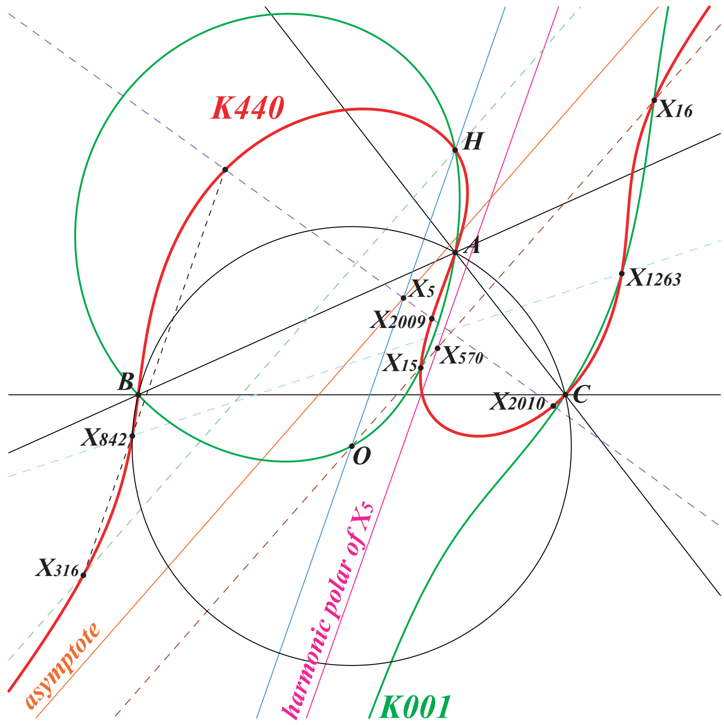

Figure 14: The cubic **K440**

K440 contains the ETC centers X_4, X_{15}, X_{16}, X_{316}, X_{511}, X_{842}, X_{1263}, X_{2009}, X_{2010}. The lines $X_{316}X_{842}$ and $X_{2009}X_{2010}$ meet at another point of the cubic.

The pencil contains two other $\mathcal{K}^+$ with rather complicated equations.

5.4 The cubic K441

K441 contains the ETC centers X_4, X_6, X_{15}, X_{16}, X_{111}, X_{1263}, X_{2079}, X_{2165}. The infinite point is E_{507}, that of the line X_5X_6 which is the orthic line of the cubic. See figure 15.

The singular focus is the second intersection of the line $X_{114}X_4$ with the circle cited above.

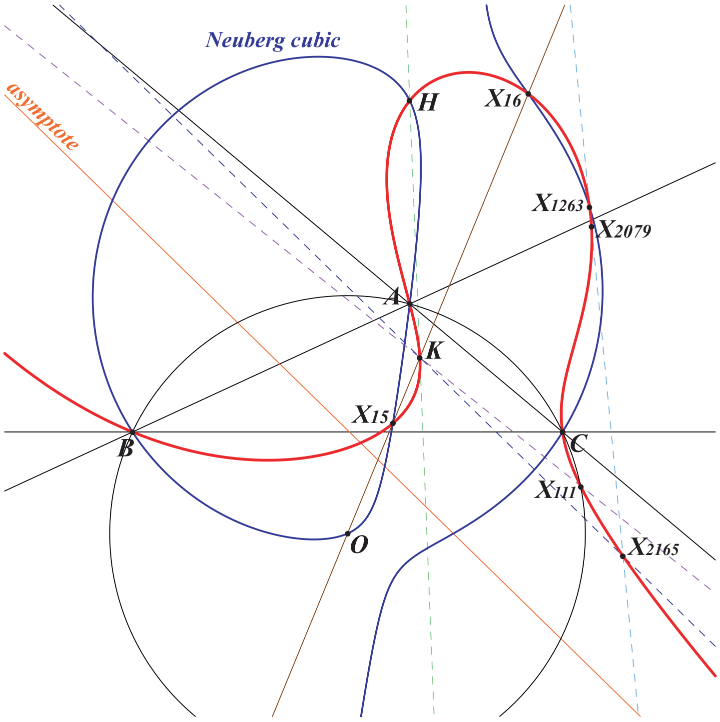

Figure 15: The cubic **K441**

5.5 Summary and other cubics

The following table summarizes the mentioned cubics of the pencil $\mathcal{N}$ according to the third point P on the Brocard axis and presents several other cubics of least interest. Their common points X_4, X_{15}, X_{16}, X_{1263} are not repeated in the table.

Notes :

– E_{216} lies on the lines X_3X_6, X_5X_{195}, $X_{49}X_{51}$, $X_{54}X_{143}$, etc.

Table 3: Cubics of the Neuberg pencil

P	cubic / centers on the cubic	F	remarks
X_3	**K001** Neuberg cubic	X_{110}	$p\mathcal{K}$
X_6	**K441**		
X_{32}	X_{32}, X_{729}		
X_{39}	X_{39}, X_{755}, X_{2782}		
X_{52}	**K050** Neuberg orthic cubic	X_{137}	$p\mathcal{K}$
X_{58}	X_{58}, X_{106}		
X_{61}	X_{61}, X_{2380}		
X_{62}	X_{62}, X_{2381}		
X_{182}	X_{98}, X_{182}, X_{1503}, X_{2980}		
X_{187}	X_{187}, X_{843}		
X_{216}	X_{53}, X_{216}		
X_{284}	X_{284}, X_{2291}		
X_{511}	**K440**	X_5	$\mathcal{K}^+$
X_{567}	X_{567}, X_{1141}	X_{23}	
X_{568}	X_{265}, X_{568}		
X_{575}	X_{23}, X_{542}, X_{549}, X_{575}		
X_{576}	**K304**		
X_{970}	X_{517}, X_{970}		
X_{1326}	X_{1326}, X_{2712}		
X_{1333}	X_{739}, X_{1333}		
X_{1350}	X_{1297}, X_{1350}		
X_{2080}	X_{2080}, X_{2698}		
X_{2420}	X_{112}, X_{2420}		
X_{3095}	X_{327}, X_{3095}		
E_{216}	**K439**		$p\mathcal{K}$
?	none	X_{114}	

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