A Remarkable Rational Transformation Related with Pivotal Cubics

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Abstract

We study a rational transformation closely related with several special kinds of pivotal cubics in particular stelloids, orthopivotal cubics and axial cubics. This brings a connexion between several apparently unrelated properties of pivotal cubics and shows the strong implication of the Euler-Morley quintic Q003.

Preamble

In this paper, we use the barycentric product and barycentric quotient of two points M, N denoted $M \times N$ and $M \div N$ and also the barycentric square M^2 of a point M. We denote by M/N the cevian quotient of M and N (or $M-$Ceva conjugate of N) i.e. the perspector of the cevian triangle of M and the anticevian triangle of N. We hereby recall some definitions.

1. Let $\gamma(M)$ be the conic passing through M, the vertices of the anticevian triangle of M and N/M. Define $\gamma(N)$ similarly. These conics are both tangent to the line MN assuming that M and N are distinct. In this case, the barycentric product $M \times N$ is the intersection of the polars $\lambda(M)$, $\lambda(N)$ of the centroid G of ABC in these two conics.

   With $M = p : q : r$ and $N = u : v : w$, the equations of $\gamma(M)$ and $\lambda(M)$ are respectively

   $$\sum_{\text{cyclic}} qr(rv - qw)x^2 = 0 \quad \text{and} \quad \sum_{\text{cyclic}} qr(rv - qw)x = 0.$$

   Note that the intersection of the polars of a point P is the isoconjugate P^* of P in the isoconjugation which swaps M and N i.e. the isoconjugation with pole $\Omega = M \times N$.

2. The barycentric quotient $M \div N$ is the barycentric product of M and its isotomic conjugate tM.

3. The barycentric square M^2 is the pole of G in the pencil of conics passing through M and the vertices of the anticevian triangle of M. It is also the intersection of the lines G, tcM and M, ctM where "c" denotes the complement of a point. Note that the line M, ctM also contains G/M. It is in fact the line passing through the centers of the inconic and circumconic with same perspector M.

4. Remark : the barycentric cube M^3 of M is the pole of tM in the pencil of conics above.

We can see that these notions are entirely affine.

1 The transformation φ

1.1 Definition

In the plane of the reference triangle ABC, let M be a point with barycentric coordinates $x : y : z$. We define the rational transformation φ by

$$\varphi : M = x : y : z \mapsto N = \frac{S_B\, y - S_C\, z}{c^2\, y^2 - b^2\, z^2} : \cdots : \cdots \tag{1}$$

where $S_A = (b^2 + c^2 - a^2)/2$ and similarly S_B, S_C as usual.

Equivalently, the coordinates of N can be written under the form

$$N = \varphi_1(x, y, z) : \varphi_2(x, y, z) : \varphi_3(x, y, z)$$

where

$$\varphi_1(x, y, z) = (S_B\, y - S_C\, z)(b^2\, x^2 - a^2\, y^2)(c^2\, x^2 - a^2\, z^2), \tag{2}$$

and $\varphi_2(x, y, z)$, $\varphi_3(x, y, z)$ likewise.

We immediately recognize the equation of the altitude AH namely $S_B\, y - S_C\, z = 0$ and that of the two bisectors at A namely $c^2\, y^2 - b^2\, z^2 = 0$.

1.2 Constructions

In general, for a given point M, the construction of N can be realized as follows.

Let $L(M)$ be the line through M and the orthocenter H of ABC. Let $H(M)$ be the rectangular diagonal hyperbola passing through M and the in/excenters of ABC. With $M = u : v : w$ the equations of $L(M)$ and $H(M)$ are

$$\sum_{\text{cyclic}} S_A(S_B\, v - S_C\, w)\, x = 0 \text{ and } \sum_{\text{cyclic}} (c^2 v^2 - b^2 w^2)\, x^2 = 0 \text{ respectively.}$$

If S is the pole of $L(M)$ in $H(M)$ then N is the barycentric product of H and S.

Obviously, this construction is not valid for certain so-called singular points as we will see below.

Remarks (see Figure 1)

1. If M' is the second intersection of $L(M)$ and $H(M)$, it is clear that $\varphi(M') = \varphi(M) = N$.
2. The tangent $T(M)$ at M to $H(M)$ contains S and also the isogonal conjugate M^* of M. This tangent contains a fixed point P if and only if M lies on $p\mathcal{K}(X_6, P)$, the isogonal pivotal cubic with pivot P.
3. S also lies on the polar (h) of H in $H(M)$ which also contains O, $M^2 \div H$ and obviously the harmonic conjugate H' of H with respect to M and M'. Note also that S^* lies on $L(M)$.
4. The center ω of $H(M)$ lies on the circumcircle. Its coordinates are

   $$\frac{1}{c^2\, v^2 - b^2\, w^2} : \cdots : \cdots .$$

   ω is the trilinear pole of the line KM^2 where M^2 is the barycentric square of M. This latter line is the polar (g) of G in $H(M)$ and it also contains N.

5. In particular, $N = \varphi(M)$ and M^2 coincide if and only if M lies on the Euler-Morley quintic Q003. In other words, the restriction of φ to Q003 is the barycentric square as far as M is not a vertex or an in/excenter of ABC (see §1.3 below).

6. If M is fixed and m variable on $H(M)$ then $n = \varphi(m)$ lies on the line KM^2 therefore φ transforms any diagonal rectangular hyperbola into a line passing through K.

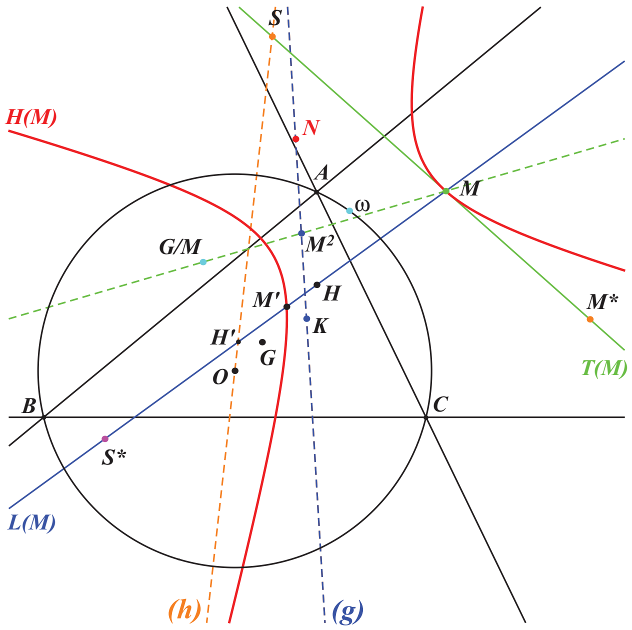

Figure 1: Construction of $N = \varphi(M)$

This construction can be easily "reversed" if we use the fact that the center of $H(M)$ is the trilinear pole of the line passing through the Lemoine point K and N.

Hence, for a given N, let us consider successively :

– the barycentric quotient $S = N \div H$,

– the isogonal conjugate S^* of S (which is in fact the pole of S in the pencil of the rectangular diagonal hyperbolas passing through the in/excenters),

– the trilinear pole ω of the line KN,

– the rectangular diagonal hyperbola $H(N)$ of the pencil with center ω i.e. passing through the reflection of one in/excenter about ω.

The intersections of $H(N)$ and the line HS^* are two points M_1, M_2 such that $\varphi(M_1) = \varphi(M_2) = N$.

These two points are not necessarily real nor distinct. Nevertheless, we already can say that φ is a $(2, 1)$ correspondence which is not generally birational unless M_1 and M_2 coincide. All this will be discussed below.

1.3 Singular points of φ and consequences

With equation (2) above, the transformation φ has apparently eight singular points namely A, B, C, H and the four in/excenters I, I_a, I_b, I_c.

This can also be obtained from the constructions above. Indeed, when $M = H$, the line $L(M)$ is not defined and when M is an in/excenter, the hyperbola $H(M)$ is not defined.

Nevertheless, we can "cover" these five points if we observe that the tangent $T(M)$ (with M not on the line at infinity) contains H if and only if M lies on the Orthocubic K006 since H, S and S^* must be collinear. This is obviously true for our five points. In such case, S and M coincide and then the images of the five points are their barycentric products with H. This will be generalized below.

It follows that $\varphi(H) = H^2 = X_{393}$ and $\varphi(X_1) = H \times X_1 = X_{19}$. The images of I_a, I_b, I_c are the extraversions of X_{19}.

Now, if $M = A$, the hyperbola $H(M)$ degenerates into the bisectors at A and the line $L(M)$ is the altitude AH hence $S = A$. In fact, when M lies on these bisectors, we always have $S = A$ and therefore $N = A$.

The transformation φ generally transforms a curve $\mathcal{C}$ of degree n into a curve $\mathcal{C}'$ of degree $5n$ but this degree may reduce under certains conditions on the so-called "fundamental points" of φ which might lie on $\mathcal{C}$. On the other hand, $\mathcal{C}'$ may split into several curves of lower degree. We shall follow Coolidge, see [1], in the next paragraphs to be more explicit.

Before doing so, we recall in a simple langage these notions of "fundamental points" and "fundamental curves" when applied to φ.

In general and as already said, φ maps two distinct points M_1, M_2 as defined above onto one single point N but, in some cases, a point N can "explode" into a set of points M which can be a line or several lines or even a curve of degree m. N is then said to be a fundamental point of order m and the curve is the fundamental curve associated with N.

If M_1 and M_2 be coincident (at M) for some point N then φ becomes a $(1, 1)$ correspondence and actually a birational transformation sometimes called a Cremona transformation. The locus of M is then the locus of the double points of φ.

All these special curves are easily obtained with the Jacobian of φ.

1.4 Jacobian of the transformation φ

We express that the polar lines of a point P in the three curves with equations $\varphi_i(x, y, z) = 0, (i = 1, 2, 3)$, are concurrent and we obtain that P must lie on a curve called the Jacobian of the net generated by these three curves.

When we compute the Jacobian of φ , we find a curve of degree 12 which factorizes into :

– the six bisectors of ABC,
– the McCay cubic K003, pivotal isogonal cubic with pivot O,
– the Orthocubic K006, pivotal isogonal cubic with pivot H.

First recall the results proven above.

Proposition 1 *The transformation φ maps any point (different of A) which lies on the two bisectors at A onto the point A.*

In other words, φ explodes A into the two bisectors at A or, equivalently contracts these bisectors at A.

When the non-singular point M lies on K003, the point M^* (hence different of M) also lies on K003 and on the line OM hence the tangent at M to $H(M)$ must contain O. This is in fact true for any pivotal isogonal cubic with pivot P instead of O.

The four points O, M, M^*, S being collinear, their isogonal conjugates H, M, M^*, S^* must lie on a same circum-conic which is a rectangular hyperbola. But we know that H, M, S^* are collinear therefore H and S^* must coincide hence $S = O$ and finally $N = O \times H = K$. This gives the following

Proposition 2 *The transformation φ maps any non-singular point on the McCay cubic onto the Lemoine point K.*

In other words, φ explodes K into the McCay cubic K003 or, equivalently contracts K003 at K.

Now, if we take the non-singular point M on K006 so that H, M and M^* are collinear, the line HM is $L(M)$ and the line MM^* is the tangent at M to $H(M)$ hence $M' = M$ and $M = S$. From this, we obtain

Proposition 3 *The transformation φ is a birational transformation when it is restricted to the Orthocubic.*

The barycentric multiplication by H preserves alignment of points and degree of curves hence if H, $M = S$ and $M^* = S^*$ are collinear on K006, the points $H^2 = X_{393}$, $N = S \times H$ and $N' = S^* \times H$ are collinear on a circum-cubic. Let us rewrite N' under the form $N' = H \times K \div S = K \times H^2 \div N$ to see that X_{393}, N and $K \times H^2 \div N$ are collinear which is exactly the condition for N to lie on the pivotal cubic with pivot X_{393} and isopivot $K = X_6$. This is the cubic K678 in [3]. We conclude with

Proposition 4 *The transformation φ maps the Orthocubic K006 onto the cubic K678 $= p\mathcal{K}(X_6 \times X_{393}, X_{393})$. Furthermore, K678 can be considered as the barycentric product of the Orthocubic K006 and H.*

1.5 Fixed points of φ

Proposition 5 *The transformation φ has one and only one fixed point, namely the centroid G of ABC.*

This is easily obtained from a computation but we can also give a geometric explanation.

M is a fixed point if and only it is a non-singular point such that $M = N = H \times S$.

K, M, M^2 are collinear hence, by barycentric quotient with M, the points $K \div M = M^*$, G, M are also collinear therefore M must lie on the Thomson cubic K002 $= p\mathcal{K}(X_6, X_2)$.

M, H, S^* are collinear hence, since $S^* = K \div (M \div H) = X_{25} \div M$, the points H, M, $X_{25} \div M$ are also collinear therefore M must lie on the cubic X233 $= p\mathcal{K}(X_{25}, X_4)$.

K002 and X233 meet at A, B, C with the same tangents (since they have the same isopivot namely K) and therefore at three other points : H, K which must be excluded for the reasons given above and finally G.

1.6 Involutive φ

A straightforward computation easily gives

Proposition 6 *The transformation φ is an involution when it is restricted to the Thomson cubic K002 and, in this case, it coincides with the $G-$Ceva conjugation.*

In other terms, φ maps any non-singular point M on K002 onto the center (resp. perspector) of the circum-conic with perspector (resp. center) M.

Thus, φ leaves K002 unchanged and two isogonal conjugate points M, M^* of K002 (hence collinear with G) are transformed into two points N, N' collinear with H. Note that the lines MN and M^*N' intersect at K, the isopivot of K002.

At last, the pole S of the line HNN' in the diagonal rectangular hyperbola passing through N lies on the cubic K099 $= p\mathcal{K}(X_{394}, X_{69})$ therefore the barycentric product of K099 and H is the Thomson cubic.

2 Images of certain curves under φ

Let $\mathcal{C}$ be a curve of degree n and let $\mathcal{C}'$ be its transform under φ. Recall that the degree of $\mathcal{C}'$ is $5n$ but $\mathcal{C}'$ can split into several curves of lower degree.

2.1 Images of lines

If $\mathcal{C}$ is a line, say $\mathcal{L}$, then $\mathcal{C}'$ is in general a proper (undecomposed) quintic passing through A, B, C which are nodes (since $\mathcal{C}$ meets each pair of bisectors at two points which are mapped onto a vertex of ABC) and K which is a triple point (since the line meets the McCay cubic at three points which are mapped onto K).

It follows that $\mathcal{C}'$ has one remaining common point with the sideline BC namely the image of the intersection of $\mathcal{L}$ and the altitude AH. This is obvious with equation(1) above.

Furthermore, $\mathcal{L}$ meets

– the Orthocubic K006 at three points and then $\mathcal{C}'$ must contain their barycentric products with H,

– the Thomson cubic K002 at three points and then $\mathcal{C}'$ must contain their $G-$Ceva conjugates.

At last, note that $\mathcal{C}'$ meets each symmedian of ABC at five known points namely K thrice and the corresponding vertex of ABC twice.

For example, the line at infinity is transformed into the proper quintic Q053.

Thus, if $\mathcal{C}'$ contains another point on a symmedian then it must split into this symmedian and a (possibly decomposed) circum-quartic passing through K with nodes at two vertices of ABC.

From this, we easily obtain :

1. if $\mathcal{C}$ is a line passing through A then $\mathcal{C}'$ splits into the symmedian AK and a quartic with nodes B, C.

2. if $\mathcal{C}$ is the sideline BC then $\mathcal{C}'$ splits into the symmedians BK, CK and a cubic with node A.

3. if $\mathcal{C}$ is a line passing through H then $\mathcal{C}'$ splits into the line HK and a circum-conic (counted twice) passing through K. Note that $\mathcal{C}$ and the conic intersect on K233 $= p\mathcal{K}(X_{25}, X_4)$. For example, the Euler line gives the circum-conic passing through G, K and the line HK gives the Jerabek hyperbola.

4. in particular, if $\mathcal{C}$ is an altitude of ABC then $\mathcal{C}'$ splits into the opposite sideline (counted twice), the symmedian passing through the same vertex (counted twice) and the line HK.

5. if $\mathcal{C}$ is a line passing through X_1 then $\mathcal{C}'$ splits into the circum-conic passing through K, X_9, X_{19}, etc, and a circum-cubic.

6. if $\mathcal{C}$ is the line passing through X_1 and H then $\mathcal{C}'$ splits into the line HK and the circum-conic above counted twice.

2.2 Images of conics

In general, φ transforms a conic into a curve of degree 10.

If we discard the "parasite" lines related with φ, we have the following results :

– if $\mathcal{C}$ is a circum-conic of ABC then $\mathcal{C}'$ is generally a septic,

– if $\mathcal{C}$ is a rectangular circum-hyperbola of ABC then $\mathcal{C}'$ is generally a sextic,

– if $\mathcal{C}$ is a diagonal conic of ABC passing through the in/excenters then $\mathcal{C}'$ is generally a conic,

– if $\mathcal{C}$ is the diagonal conic of ABC passing through the in/excenters and H then $\mathcal{C}'$ is the line HK.

2.3 Images of cubics

The general case has very little interest since φ transforms a cubic into a curve of degree 15.

We will simply focus on the cubics passing through the eight singular points which are in fact the isogonal pivotal cubics of the Euler pencil i.e. the cubics with pivot on the Euler line. These cubics also contain O. Recall that the McCay cubic K003 is contracted at K and is excluded in the sequel.

If we consider two pivots P_1, P_2 harmonically conjugated with regard to O and H and defined by $\overrightarrow{HP} = \dfrac{2}{2 \pm \lambda}\,\overrightarrow{HO}$, we will have $\varphi(P_1) = \varphi(P_2) = Q$ hence the two corresponding cubics $p\mathcal{K}(K, P_1)$, $p\mathcal{K}(K, P_2)$ are transformed into the same cubic namely $p\mathcal{K}(K \times Q, Q)$.

The pivot Q lies on the circum-conic passing through G, K and the pole $K \times Q$ lies on the circum-conic passing through K, X_{32}.

For example,

– the Neuberg and Napoleon cubics K001, K005 are transformed into K095,

– the Thomson and Darboux cubics K002, K004 are transformed into K002.

Recall that the "lonesome" Orthocubic K006 is transformed into K678 since φ is here birational.

We shall meet these cubics again below in §4.2.

3 φ in relation with special pivotal cubics

We first recall results from [2], §3.5.3, §6.4.2 and [4], §6.2.1.

3.1 φ and the pivotal cubics with asymptotes concurring at G

In the particular case of P on the Thomson cubic K002, we know that φ is involutive and coincide with $G-$Ceva conjugation.

Hence, for any pivot P on the Thomson cubic K002, the pivotal cubic $p\mathcal{K}(\varphi(P), P) = p\mathcal{K}(G/P, P)$ has three real asymptotes concurring at G. This is actually true even if P is not on K002 but, in our case, the pole $\Omega = \varphi(P)$ also lies on K002 and the isopivot is the anticomplement of P, a point on the Lucas cubic K007.

3.2 φ and the pivotal stelloids

For any pivot P on the Neuberg cubic K001, the pivotal cubic $p\mathcal{K}(\varphi(P), P)$ is a stelloid with

– pole $\Omega = \varphi(P)$ on K095 $= \varphi($K001$)$,

– isopivot P^* on K060,

– radial center X on Q004.

Recall that a pivotal cubic has its asymptotes parallel to those of the McCay cubic K003 if and only if its pivot lies on the cubic K080.

3.3 φ and the pivotal orthopivotal cubics

For any P on the Napoleon cubic K005, the orthopivotal cubic $\mathcal{O}(P)$ is pivotal with
– pole $\Omega = \varphi(P)$ on K095 $= \varphi($K001$)$,
– pivot Q on K060 and on the line $X_5\,P$, Q being the perspector of ABC and the -2-pedal triangle of P i.e. the image of the pedal triangle of P under the homothety $h(P,-2)$.
– isopivot Q^* on K001, Q^* being the inverse (in the circumcircle) of the isogonal conjugate of Q.

These pivotal cubics are circular and contain the Fermat points X_{13}, X_{14}. See for example K001 $= \mathcal{O}(X_3)$, K058 $= \mathcal{O}(X_1)$, K059 $= \mathcal{O}(X_4)$, K060 $= \mathcal{O}(X_5)$.

3.4 φ and the pivotal axial cubics

If P is the infinite point of a line L, there are two (not always real) pivotal axial cubics having their axes of symmetry perpendicular to L.
– the pole $\Omega = \varphi(P)$ lies on Q053, the φ image of the line at infinity,
– their pivots lie on the line PP^* where $P^* = \varphi(P)$.

See the cubics K335 ($P = X_{2393}$, axis $X_3\,X_{647}$) and K528 ($P = X_{30}$, axis $X_3\,X_{523}$) for example.

3.5 φ and the pivotal cubics through the I_x-AntiCevian points

See Table 23 in [3] for detailed explanations. We simply recall that a point is said to be an I_x-AntiCevian point when it is an in/excenter of its anticevian triangle. There are four (not always real) such points which all lie on three cubics denoted $\mathcal{K}(A)$, $\mathcal{K}(B)$, $\mathcal{K}(C)$ and therefore on any cubic $\mathcal{K}(Q = u : v : w) = u\,\mathcal{K}(A) + v\,\mathcal{K}(B) + w\,\mathcal{K}(C)$. The equation of $\mathcal{K}(A)$ is

$$x\,(c^2\,y^2 - b^2\,z^2) - 2\,(S_B\,y - S_C\,z)yz = 0.$$

For any point Q on the Neuberg cubic K001, the cubic $\mathcal{K}(Q)$ is a pivotal cubic with
– pole $\Omega = \varphi(Q)$ on K095 $= \varphi($K001$)$,
– pivot P on K060, P being the perspector of ABC and the triangle formed by the reflections of Q in the sidelines of ABC,
– isopivot P^* on K005 where $P^* = \Omega \div P$.

3.6 φ and the pivotal cubics through the CPCC points

See Table 11 in [3] for detailed explanations. We simply recall that a point is said to be a CPCC point when its pedal and cevian circles coincide or, equivalently, it is the orthocenter of its cevian triangle. Apart the trivial case H, there are four (not always real) such points which all lie on three nodal cubics denoted $\mathcal{K}a$, $\mathcal{K}b$, $\mathcal{K}c$ and therefore on any so-called Orion cubic $\mathcal{K}(Q = u : v : w) = u\,\mathcal{K}a + v\,\mathcal{K}b + w\,\mathcal{K}c$. The equation of $\mathcal{K}a$ is

$$x\,(c^2\,y^2 - b^2\,z^2) + (S_B\,y - S_C\,z)yz = 0.$$

Note that $\mathcal{K}(Q = G)$ is the Lucas cubic K007 and $\mathcal{K}(Q = O)$ is the Darboux cubic K004.

For any point Q on the Thomson cubic K002, the cubic $\mathcal{K}(Q)$ is a pivotal cubic with
– pole $\Omega = \varphi(Q) = G/Q$ on K002,
– pivot P on the Lucas cubic K007 since P is the anticomplement of the isogonal conjugate of Q,
– isopivot P^* on K004.

3.7 φ and the pivotal Thomson centroidal cubics

See CL040 in [3] for detailed explanations. A Thomson centroidal cubic denoted by $TC(Q = u : v : w)$ is defined as above with three cubics $TC(A)$, $TC(B)$, $TC(C)$ where the equation of $TC(A)$ is

$$x\,(c^2\,y^2 - b^2\,z^2) - (S_B\,y - S_C\,z)yz = 0.$$

$TC(Q)$ always contains G and the infinite points of the Thomson cubic K002. It is a pivotal cubic if and only if Q lies on the Darboux cubic K004 in which case
– its pole $\Omega = \varphi(Q) = G/Q$ lies on K002,
– its pivot P lies on the Lucas cubic K007,
– its isopivot P^* lies on K002.

4 Generalization

4.1 The cubics $\mathcal{K}_\lambda(Q)$

Looking back at the equations of $\mathcal{K}(A)$, $\mathcal{K}a$ and $TC(A)$ given above, it seems relevant to consider the equation of a more general cubic $\mathcal{K}_\lambda(A)$ under the form

$$x\,(c^2\,y^2 - b^2\,z^2) + \lambda\,(S_B\,y - S_C\,z)yz = 0,\ \lambda \in \mathbb{R} \cup \{\infty\}$$

and, consequently, the corresponding cubic defined as above

$$\mathcal{K}_\lambda(Q = u : v : w) = u\,\mathcal{K}_\lambda(A) + v\,\mathcal{K}_\lambda(B) + w\,\mathcal{K}_\lambda(C).$$

$\mathcal{K}_\lambda(A)$ is a nodal circum-cubic with node A. Its nodal tangents are the two bisectors at A. Its isogonal transform is a conic $\mathcal{K}^*_\lambda(A) = \mathcal{C}_\lambda(A)$ with equation

$$a^2(c^2\,y^2 - b^2\,z^2) + \lambda\,x\,(c^2 S_C\,y - b^2 S_B\,z) = 0.$$

For any λ, the conic $\mathcal{C}_\lambda(A)$ contains A with a tangent passing through O and the feet of the bisectors at A on the sideline BC.

Remark :
– when $\lambda = 0$, $\mathcal{C}_\lambda(A)$ decomposes into the bisectors at A,
– when $\lambda = \infty$, $\mathcal{C}_\lambda(A)$ decomposes into the lines BC and AO,
– when $\lambda = 2$, $\mathcal{C}_\lambda(A)$ is the $A-$Apollonius circle.

The sum of the three conics $\mathcal{C}_\lambda(A)$, $\mathcal{C}_\lambda(B)$, $\mathcal{C}_\lambda(C)$ identically vanishes for every λ hence these conics are in a same pencil and contain four (real or not) points denoted $Q_i, i \in \{1, 2, 3, 4\}$.

It follows that, for a given λ, all the cubics $\mathcal{K}_\lambda(Q)$ contain (apart A, B, C) four fixed points P_i which are obviously the isogonal conjugates of the common points Q_i of the three conics.

In other words, each value of λ corresponds to a set of four points P_i which lie on $\mathcal{K}_\lambda(Q)$ for any Q.

For example, with $\lambda = -1$, the points on the conics are K and the vertices of the Thomson triangle hence $\mathcal{K}_{-1}(Q) = TC(Q)$ contains G and the infinite points of the Thomson cubic K002 as seen above in §3.7.

4.2 A connexion with the Euler-Morley curves Q003 and Q002

When we eliminate λ between the equations of $\mathcal{K}_\lambda(A)$, $\mathcal{K}_\lambda(B)$, $\mathcal{K}_\lambda(C)$ we obtain the locus of the four common points P_i when λ varies and this locus turns out to be the Euler-Morley quintic Q003. It follows that the points Q_i lie on the Euler-Morley quartic Q002.

It is also easy to obtain this latter result if we remark that

$$\sum_{\text{cyclic}} a^2 S_A\, yz\, \mathcal{C}_\lambda(A) = 0$$

is independent of λ and is actually the equation of Q002.

Recall that the equations of Q003 and Q002 are

$$\sum_{\text{cyclic}} a^2y^2z^2(S_B\, y - S_C\, z) = 0 \text{ and } \sum_{\text{cyclic}} a^4 S_A(c^2\, y^2 - b^2\, z^2)yz = 0 \text{ respectively.}$$

See figure 2 and [3] for more details about these two extraordinary curves having lots of other properties.

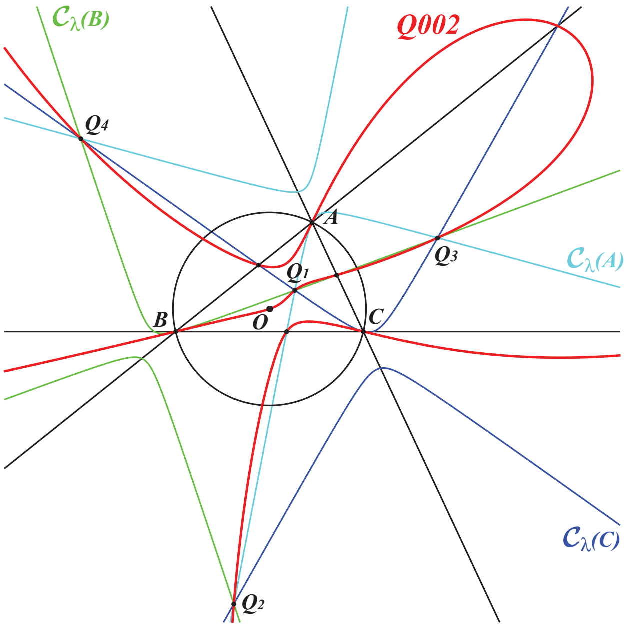

Figure 2: Q002 and the conics $\mathcal{C}_\lambda(A)$, $\mathcal{C}_\lambda(B)$, $\mathcal{C}_\lambda(C)$

Hence $\mathcal{K}_\lambda(A)$ and Q003 meet at 15 points namely A counted 6 times (since the two curves have a node at A with the same tangents), B and C each counted twice, the foot of the altitude AH on BC and our four points P_i.

Thus we see that the points on Q003 are distributed in groups of four and so are the points on Q002. Another characterization of this property is given in [3] at the page Q002. It is connected with the pivotal cubics of the Euler pencil and with hyperbolas homothetic to the Jerabek hyperbola and passing through O and K.

These properties can also be used to realize the construction of the Euler-Morley curves. See the page Q002 in [3].

4.3 Pivotal cubics $\mathcal{K}_\lambda(Q)$

This cubic $\mathcal{K}_\lambda(Q)$ is a pivotal cubic if and only if Q lies on the isogonal pivotal cubic with pivot F on the Euler line such that $\overrightarrow{HF} = \dfrac{2}{2+\lambda}\,\overrightarrow{HO}$.

With $\lambda = -2$, we have $F = X_{30}$ and then Q must lie on the Neuberg cubic K001 as in §3.5. With $\lambda = 1$, we have $F = X_2$ and then Q must lie on the Thomson cubic K002 as in §3.6. With $\lambda = -1$, we have $F = X_{20}$ and then Q must lie on the Darboux cubic K004 as in §3.7.

Now, for a given λ or a given F, the pivotal cubic $\mathcal{K}_\lambda(Q)$ with Q on $p\mathcal{K}(K, F)$ has its

– pivot P on $p\mathcal{K}(\varphi(F), \varphi(F) \div H)$,

– isopivot P^* on the cubic $p\mathcal{K}(K, F')$ where F' is defined by $\overrightarrow{HF'} = \dfrac{2}{2-\lambda}\overrightarrow{HO}$,

– pole Ω on $p\mathcal{K}(K \times \varphi(F), \varphi(F))$.

Note that $\varphi(F) = \varphi(F')$ hence the locus of the poles and the locus of the pivots of the two pivotal cubics $\mathcal{K}_\lambda(Q)$ and $\mathcal{K}_{-\lambda}(Q)$ are the same.

For instance, in §3.3 ($\lambda = 2$) and §3.5 ($\lambda = -2$) these loci are K095 and K060 respectively.

This is related with Pinkernell's paper [7] where these cubics are called d-pedal and d-cevian cubics. Indeed, if $P_aP_bP_c$ is the pedal triangle of P, let us denote by $Q_a^+Q_b^+Q_c^+$ and $Q_a^-Q_b^-Q_c^-$ the images of $P_aP_bP_c$ under the homotheties $h(P, \lambda)$ and $h(P, -\lambda)$ respectively.

ABC and $Q_a^+Q_b^+Q_c^+$ (resp.$Q_a^-Q_b^-Q_c^-$) are perspective if and only if P lies on $p\mathcal{K}(K, F')$ (resp. $p\mathcal{K}(K, F)$) and, in both cases, the locus of the perspector is $p\mathcal{K}(\varphi(F), \varphi(F) \div H)$.

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