Two Related Transformations and Associated Cubics

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Created : April 1, 2014
Last update : April 6, 2014

Abstract

We study a transformation closely related with the orthocorrespondence and consequently a family of circum-cubics similar to orthopivotal cubics. All these cubics contain the four \( Ix \)-anticevian points.

1 The two transformations \( \Phi \) and \( \Psi \)

1.1 Definition

The orthocorrespondence \( \Phi \) is defined and studied in [6]. We simply recall that it maps a point \( M = x : y : z \) to its orthocorrespondent \( M^\perp \) with barycentric coordinates

\[
x (-S_A x + S_B y + S_C z) + a^2 yz : \cdots : \cdots .
\]

(1)

Now, let us consider the transformation \( \Psi \) that maps \( M \) to its isogonal conjugate \( M^\bullet \) with regard to the anticevian triangle of \( M \). A computation easily gives the coordinates of \( M^\bullet \) namely

\[
x (-S_A x + S_B y + S_C z) - a^2 yz : \cdots : \cdots .
\]

(2)

These coordinates are surprisingly almost identical. Furthermore, by addition and substraction, we find that these two points lie on a same line that contains the isogonal conjugate \( M^* \) of \( M \) (with respect to the usual reference triangle \( ABC \)) and also the cevian quotient \( H/M \) (the perspector of the orthic triangle \( H_aH_bH_c \) and the anticevian triangle of \( M \) also called \( H-Ceva \) conjugate of \( M \)). Indeed, \( M^* = a^2 yz : \cdots : \cdots \) and \( H/M = x (-S_A x + S_B y + S_C z) : \cdots : \cdots . \)

The points \( M^\bullet \) and \( M^\perp \) are clearly harmonic conjugates with respect to \( M^* \) and \( H/M \).

The line passing through these four points does not contain \( M \) itself unless it lies on the Euler-Morley quintic \( Q003 \), a curve with many interesting properties.

1.2 Properties of the transformation \( \Psi \)

Singular points

When the coordinates of \( M^\bullet \) are equated to zero, we obtain the equations of three conics \( \sigma_1, \sigma_2, \sigma_3 \) which generally have no common points.

\( \sigma_1 \) contains \( B, C, H_b, H_c \) and its tangents at \( B, C \) pass through the midpoint of the altitude \( AH \).

\( \sigma_2 \) and \( \sigma_3 \) meet at \( A, H_a \) and two imaginary conjugate points but none of these points lies on \( \sigma_1 \). See figure 1.

It follows that \( \Psi \) has no singular point hence it always transforms a curve of degree \( n \) into a curve of degree \( 2n \).

Fixed points

When we express that \( M \) and \( M^\bullet \) coincide, we find three nodal circum-cubics with nodes at \( A, B, C \) hence these three latter points are fixed points.
On the other hand, \( M \) (distinct of \( A, B, C \)) and \( M^* \) coincide if and only if \( M \) is an in/excenter of its anticevian triangle i.e. \( M \) is one of the four \( Ix \)-anticevian points of Table 23. These points lie on the rectangular hyperbola passing through \( X_5, X_6, X_52, X_{195}, X_{265}, X_{382}, X_{2574}, X_{2575} \).

From this, we conclude that \( \Psi \) has seven fixed points which are the common points of several known cubics such as \( K005, K049, K060 \), etc. See §3.1 below.

1.3 Images of some triangle centers under \( \Psi \)

Angel Montesdeoca has kindly provided a list of ETC pairs (until \( X_{5550} \)) of the form \((X, \Psi(X)):\)

\((X_1, X_{40}), (X_2, X_{69}), (X_4, X_{20}), (X_6, X_{22}), (X_{30}, X_{146}), (X_{511}, X_{147}), (X_{512}, X_{148}), (X_{513}, X_{149}), (X_{514}, X_{150}), (X_{515}, X_{151}), (X_{516}, X_{152}), (X_{517}, X_{153}), (X_{523}, X_{3448})\).

Surprisingly, the crop was poor and most of the points are images of points \( X \) at infinity. See below.

1.4 Images of lines under \( \Psi \)

\( \Psi \) transforms any line \( L \) into a conic \( C \) and, obviously, any line passing through one (or two) fixed point(s) into a conic passing through this (these) same fixed point(s).

Image of the line at infinity \( L_\infty \)

It is easy to verify that \( \Psi \) swaps the circular points at infinity hence \( \Psi \) transforms \( L_\infty \) into a circle which turns out to be the circle with center \( H \), radius \( 2R \) i.e. the anticomplement of the circumcircle.

1.5 The inverse transformation \( \Psi^{-1} \) of \( \Psi \)

The points \( \Psi(M) \) and \( P \) coincide if and only if \( M \) is the intersection of three conics \( \gamma_A, \gamma_B, \gamma_C \) belonging to a same pencil hence having four common points but these
points are not necessarily real.

The conic $\gamma_A$ can be constructed easily since it contains five known points namely:
- $A$, $H_a$, the trace $U$ on $BC$ of the trilinear polar $\mathcal{L}(P)$ of $P$,
- the projections $A_b$, $A_c$ on $AB$, $AC$ of the intersection $P_a$ of $AP$ and $BC$.

Figure 2: Construction of $\gamma_A$

2 Related cubic curves

2.1 Definitions and equations

Let $P = p : q : r$ be a given point. Recall that the orthopivotal cubic $O(P)$ is the locus of $M$ such that $P$, $M$ and its orthocorrespondent $M^\perp = \Phi(M)$ are collinear. Any orthopivotal cubic $O(P)$ passes through seven fixed points namely $A$, $B$, $C$, the circular points at infinity and the Fermat points $X_{13}$, $X_{14}$.

We define the “anticevian isogonal” cubic $A(P)$ similarly: $A(P)$ is the locus of $M$ such that $P$, $M$ and $M^* = \Psi(M)$ are collinear.

The barycentric equation of the orthopivotal cubic $O(P)$ is

$$\sum_{\text{cyclic}} 2p(S_B y - S_C z)y z - \sum_{\text{cyclic}} a^2 (r y - q z)y z = 0 \quad (3)$$

and that of the anticevian isogonal cubic $A(P)$ is

$$\sum_{\text{cyclic}} 2p(S_B y - S_C z)y z + \sum_{\text{cyclic}} a^2 (r y - q z)y z = 0 \quad (4)$$

where

$$\sum_{\text{cyclic}} 2p(S_B y - S_C z)y z = 0 \quad (5)$$

is the equation of $pK(H \times P, P)$, the pivotal cubic with pivot $H$ and isopivot $P$, and

$$\sum_{\text{cyclic}} a^2 (r y - q z)y z = 0 \quad (6)$$
is the equation of \( pK(X_6, P) \), the isogonal pivotal cubic with pivot \( P \).

From these equations, we can see that \( O(P) \) and \( A(P) \) belong to a same pencil of cubics generated by the two pivotal cubics above which also contains a third (rather complicated) pivotal cubic.

All these cubics \( A(P) \) are therefore of type \( K_0 \) i.e. cubics without term in \( xyz \) sometimes called “apolar cubics”. See figure 3.

Note that, when a point \( M \) is common to the four cubics (as in figure 3), then six points must be collinear namely \( P, M, M^*, H/M, M^\perp \) and \( M^\perp \). Hence, \( M \) must be a point on the Euler-Morley quintic \( Q003 \). In other words, the four cubics of the pencil pass through \( A, B, C, P \) and five other points on \( Q003 \).

2.2 Immediate properties of \( A(P) \)

From the definition above, we directly obtain that \( A(P) \) must contain the seven fixed points of \( \Psi \). In other words, \( A(P) \) is a circum-cubic that passes through the four \( Ix \)-anticevian points.

\( A(P) \) must also contain \( P \) and the four (real or not) pre-images of \( P \) i.e. the points \( M \) such that \( \Psi(M) = P \).

\( A(P) \) meets the sidelines of \( ABC \) again at three points \( U, V, W \). This point \( U \) is the intersection of \( BC \) with the line passing through \( P \) and the reflection \( A' \) of \( A \) about \( BC \) with coordinates \( 0 : 2S_C p + a^2 q : 2S_B p + a^2 r \).

\( U \) is undefined when \( P = A' \) since \( A(A') \) splits into the line \( BC \) and a conic.

2.3 Construction of \( A(P) \)

Let us consider the pencil \( F \) of conics containing \( \gamma_A, \gamma_B, \gamma_C \) as in §1.5 and let \( P \) be a fixed point.

If \( L_P \) is a variable line passing through \( P \), let \( C_P \) be the conic of the pencil \( F \) that contains the anticomplement of the isogonal conjugate of the infinite point of \( L_P \). Recall that this latter point lies on the circle \( C(X_4, 2R) \).

\( L_P \) and \( C_P \) meet at two points on \( A(P) \).
2.4 Intersection of $A(P)$ with the line at infinity

The equation of $A(P)$ rewrites under the form

$$\sum_{\text{cyclic}} a^2[(p + q)y - (p + r)z]yz = (x + y + z) \sum_{\text{cyclic}} [p(b^2 - c^2) + a^2(q - r)]yz \quad (7)$$

showing that $A(P)$ meets the line at infinity at the same points as $\mathcal{p}K(X_6, cP)$, the pivotal isogonal cubic with pivot the complement $cP$ of $P$ and whose equation is the left hand member of (7).

It follows that $A(P)$ and $\mathcal{p}K(X_6, cP)$ meet at six other finite points which lie on the circum-conic $\Gamma_P$ with equation

$$\sum_{\text{cyclic}} [p(b^2 - c^2) + a^2(q - r)]yz = 0,$$

passing through the centroid $G$ of $ABC$. The perspector of this conic is the infinite point of the polar of $P$ in the circumcircle as far as $P$ is not $O$. If $P = O$, the two cubics $A(P)$ and $\mathcal{p}K(X_6, cP)$ coincide with the Napoleon cubic $K005$. See figure 4.

![Figure 4: Intersection of $A(P)$ with the line at infinity](image)

2.5 Intersection of $A(P)$ with the circumcircle of $ABC$

The equation of $A(P)$ rewrites under the form

$$\begin{align*}
(a^2yz + b^2zx + c^2xy) \sum_{\text{cyclic}} b^2c^2[p(b^2 - c^2) + a^2(q - r)]x \\
+ \sum_{\text{cyclic}} a^2[2b^2c^2p - c^2(c^2 - a^2)q + b^2(a^2 - b^2)r] \times (c^2y^2 - b^2z^2) = 0
\end{align*} \quad (8)$$

showing that $A(P)$ meets the circumcircle ($O$) at the same points as the pivotal isogonal cubic $\mathcal{p}K(X_6, Q)$ with pivot

$$Q = a^2[2b^2c^2p - c^2(c^2 - a^2)q + b^2(a^2 - b^2)r] : \cdots : \cdots .$$
If \( G_p \) is the centroid of the pedal triangle of \( P \) and if \( \cap P \) is the complement of the complement of \( P \) then \( Q \) is the homothetic of \( G_p \) under \( h(\cap P, 3) \).

It follows that \( A(P) \) and \( pK(X_6, Q) \) meet at three other (not necessarily all real) points which lie on the line \( \Lambda_P \) with equation

\[
\sum_{\text{cyclic}} b^2 c^2 (a^2 q - a^2 r + b^2 p - c^2 p) x = 0,
\]

which passes through \( X_6 \). The trilinear pole of \( \Lambda_P \) is the isogonal conjugate of the perspector of \( \Gamma_P \). Here again, \( P \) must be different of \( O \). See figure 5.

![Figure 5: Intersection of \( A(P) \) with the circumcircle of \( ABC \)](image)

3 Special cubics \( A(P) \)

3.1 Pivotal cubics \( A(P) \)

\( A(P) \) is a pivotal cubic (a \( pK \)) if and only if the tangents at \( A, B, C \) concur (at the isopivot) or, equivalently, the third points \( U, V, W \) on the sidelines of \( ABC \) form a triangle perspective with \( ABC \) (at the pivot).

Recall that \( U \) is the intersection of \( BC \) with the line passing through \( P \) and the reflection \( A' \) of \( A \) about \( BC \) hence the line \( AU \) contains the reflection \( P_a \) of \( P \) about \( BC \). The points \( P_b, P_c \) are defined likewise and then \( ABC \) and \( UVW \) are perspective if and only if \( ABC \) and \( P_aP_bP_c \) are perspective therefore if and only if \( P \) lies on the Neuberg cubic \( K001 \).

**Proposition 1** \( A(P) \) is a pivotal cubic if and only if \( P \) lies on the Neuberg cubic \( K001 \).

In this case,
- the pole lies on \( K095 \).
- the pivot lies on \( K060 \) which is \( O(X_5) \),
- the isopivot lies on the Napoleon cubic \( K005 \).

All these cubics contain the seven fixed points of \( \Psi \) as said above.
3.2 Non-pivotal cubics $\mathcal{A}(P)$

$\mathcal{A}(P)$ is a non-pivotal cubic (a $nK_0$) if and only if the points $U$, $V$, $W$ are collinear on a line whose trilinear pole is the root of the cubic.

**Proposition 2** $\mathcal{A}(P)$ is a non-pivotal cubic if and only if $P$ lies on a non-pivotal isogonal cubic with root $X_5$.

Unfortunately, this latter cubic does not contain any center of the current edition of ETC.

3.3 Cubics $\mathcal{A}(P)$ with concurring asymptotes

$\mathcal{A}(P)$ is a cubic with three real concurring asymptotes if and only if $P$ lies on the stelloid $\Sigma$ with equation

$$\sum_{\text{cyclic}} 2a^2 S_A x(c^2 y^2 - b^2 z^2) = (x + y + z) F(x, y, z)$$

(9)

where

$$F(x, y, z) = \sum_{\text{cyclic}} (b^2 - c^2)[2S_A^2 x^2 - (a^2(8S_A - a^2) + (b^2 - c^2)^2)yz].$$

The left hand member of (9) represents the McCay cubic $K003$ and $F(x, y, z) = 0$ is the equation of a rectangular hyperbola passing through $H$ and tangent at $H$ to the McCay cubic. Hence these two curves have four other common points $S_1$, $S_2$, $S_3$, $S_4$.

$\Sigma$ has its asymptotes parallel to those of the McCay cubic and concurring at the reflection of $X_{51}$ about $H$. It contains $H$, $X_{550}$, $X_{3146}$ giving the cubics $K049$, $K123$, $K127$. See figure 6.

![Figure 6: The stelloid $\Sigma$](image)

3.4 Cubics $\mathcal{A}(P)$ with $P$ on the Euler line

With $P$ on the Euler line, the cubics $\mathcal{A}(P)$ form a pencil and contain $X_4$ and $X_5$. These are the cubics $\mathcal{D}(k)$ in [4].
References


